Optimal and Efficient Mechanisms with Asymmetrically Budget Constrained Buyers*

Alexei Boulatov† Sergei Severinov‡

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Abstract

The paper characterizes the optimal (revenue-maximizing) mechanism for allocating a good to buyers who face asymmetric budget constraints. The optimal mechanism belongs to one of two classes. When the budget differences between the buyers are small, the mechanism discriminates only between high-valuation types for whom the budget constraint is binding. All low valuations buyers are treated symmetrically despite budget differences. When budget differences are sufficiently large, the mechanism discriminates in favor of buyers with small budgets when the valuations are low, and in favor of buyers with larger budgets when the valuations are high. We also provide a characterization of the constrained-efficient (surplus maximizing) mechanism and demonstrate that it shares the above properties of the optimal mechanism.

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†Higher School of Economics, ICEF, Moscow, 119049, Russia; email: aboulatov@gmail.com

‡Corresponding Author. Vancouver School of Economics, University of British Columbia, Vancouver, V6T 1L4, BC, Canada and Higher School of Economics, Moscow, Russia; email: sseverinov@gmail.com.

1 Introduction

This paper deals with mechanism design when buyers are budget constrained. Budget constraints often affect participants in trading mechanisms and institutions. In particular, consumers typically face wealth and liquidity constraints which reduce their ability to pay for the goods, especially for big-ticket items like houses and cars. In keyword search auctions at the internet search engines (Google, Bing), the advertisers typically face spending limits. Budget constraints also affect bidding and outcomes in spectrum auctions in a significant way (see e.g. Rothkopf (2007) and Bulow, Milgrom and Levin (2009)). So it is natural that budget constraints should be taken into account in the analysis and design of trading mechanisms and institutions, and there is now a growing literature exploring the implication of budgets constraints in such contexts. With some notable exceptions, discussed below, this literature focuses on the analysis of specific institutions such as different forms of auctions.

In contrast, this paper deals with the design of an optimal mechanism maximizing the seller's revenue. Since designing the most efficient mechanism is also an important objective, we provide a surplus maximizing (constrained-efficient) mechanism as an extension.

We consider a setting in which several bidders, with private values and commonly known and unequal budgets, compete for a single good. There are several real-world environments in which the bidders' budgets are typically known by the seller. First, in large-scale privatization auctions of state assets in Eastern Europe and elsewhere, and in the auctions of publicly-owned stakes in corporations or tracts of natural resources, the bidders are/were typically large corporations whose financial resources were fairly well-known, or could be estimated fairly precisely from their financial reports. Second, the sellers of high-value assets may and often do require the bidders to qualify by providing detailed financial disclosures. In a different domain, some professional sports leagues such as NHL and NFL have salary caps. So when the teams bid for players, their maximal budgets are the available room under their salary caps which is publicly known.

To focus on the effect of budget asymmetry we consider the case of identically distributed valuations. However, we show how our results generalize to an environment with asymmetrically distributed valuations in the online Appendix. An important implication of the budget asymmetry is that the designer has to construct ex-ante asymmetric allocation profiles (probability of trading and transfer function), one for each buyer. In contrast, in a symmetric situation when all budgets are equal and the bidders' valuations are identically distributed, a mechanism designer has to construct only a single allocation profile offered to every buyer. This affords a significant analytical simplification, which is not available here.

Qualitatively our optimal mechanism, which we fully characterize for the case of two bidders, belongs to one of two classes. If the budget differences between the bidders are sufficiently small, the optimal mechanism is a so-called "top-auction." In this mechanism there is a common threshold value $\bar{x}^t$ at which the budget constraint of each bidder becomes binding. All bidders with values below $\bar{x}^t$ are treated symmetrically like in a standard all-pay auction. Any bidder with a value exceeding $\bar{x}^t$ pays her budget, and gets the good with a probability that typically jumps at $\bar{x}^t$ but does not change with the bidder's value on $[\bar{x}^t, 1]$. So, all bidders with values above $\bar{x}^t$ are essentially tied. The tie-breaking rule setting the probabilities, with which different bidders with values exceeding $\bar{x}^t$ get the good, is the only instrument used by the seller to discriminate between bidders. This probability is higher for a richer bidder to compensate her for the higher payment, equal to her budget, to the seller.

The threshold $\bar{x}^t$ is determined by the sum of individual budgets and is increasing in it. In turn, $\bar{x}^t$ determines the reservation value such that types below it get the good with zero probability. This reservation value is lower than in the standard case without budget constraints, because the bidders with values above $\bar{x}^t$ pay their budgets, and the seller cannot extract more surplus from them. Therefore, the tradeoff between higher efficiency and leaving greater surplus to the bidders shifts to higher efficiency at lower values.

When the bidders' budgets are sufficiently different, the seller cannot achieve necessary differentiation between them by discriminating only "at the top" via the "top auction." In particular, it becomes impossible to allocate the good to the bidders with valuations above a common threshold $\bar{x}^t$ so that each pays her budget. So, the seller has to offer a different mechanism - a so-called "budget handicap" auction- with two kinds of discrimination between the bidders. First, she sets different thresholds. Naturally, a richer bidder faces a higher threshold. When her value exceeds this threshold, the richer bidder gets the good with a higher probability than a poorer bidder with a value above her respective threshold. This is like in the top auction, except the thresholds are now different.

Importantly, in the budget-handicap auction the seller also discriminates between buyers with low values. In particular, a poorer bidder faces a lower reservation value and also gets the good with a higher probability than a richer bidder when they both have the same value below the threshold of the poorer bidder.

Notably, our analysis also applies when only the poorer bidder faces a binding budget constraint, while the richer bidder's budget is high enough that her budget constraint is never binding. The budget-handicap auction remains optimal in this case, although its characterization becomes slightly different, as shown in Theorem 2.

The handicapping of the richer bidder in this auction creates more competition for her from a poorer one, and allows the seller to extract higher payments from the richer bidder with high values. But it also generates extra inefficiency and reduces the payments made by the richer bidder with intermediate values. Therefore, under a fixed aggregate budget, the seller is better off when the budgets are not too different and the optimal mechanism is a top auction rather than a budget-handicap auction, as shown in Theorem 3. In fact, in the top auction the seller's revenue depends only on the aggregate budget.

The optimal mechanism can be implemented via an indirect mechanism combining an all-pay auction with a lottery. Precisely, a bidder is offered a choice between buying a lottery ticket by paying her whole budget, and participating in an all pay-auction. A bidder chooses to buy a lottery ticket if her value is above her respective threshold, and participates in the all-pay auction otherwise. The difference between the top auction and the budget handicap auction is that in the former the all-pay auction is symmetric. In contrast, in the budget-handicap mechanism the all-pay auction is asymmetric and handicaps richer bidders.

In a seminal paper on optimal auctions, Myerson (1981) has considered bidders with asymmetrically distributed values and showed the optimality of handicapping the bidders whose values are more likely to be high and who, therefore, have lower virtual values.[1] In our model, the bidders' asymmetry comes from another source- budget differences. When these differences are large, an asymmetry of virtual values arises endogenously and leads to handicapping of richer bidders. While handicapping occurs at all values in Myerson (1981), in our setting only richer bidders with low values are handicapped. In contrast, richer high-value bidders get the good with a greater probability than poorer bidders with such values.

The paper closest to ours in the literature is Laffont and Robert (1996) who derive the optimal mechanism for bidders with commonly known and equal budgets. Their mechanism is symmetric and does not shed light on how the seller should treat bidders with different budgets. Yet, it is important and interesting to understand mechanism design in such ex-ante asymmetric environments as ours, since equal budgets are a knife-edge case. A surprising result of our analysis is that the bidders' threshold values at which their budget constraints become binding remain equal when budget differences are small, and the discrimination between the bidders is achieved through the probabilities of trading "at the top." So, our "top auction" provides a generalization of Laffont and Robert (1996) mechanism to a setting

[1] More recently Jehiel and Lamy (2015) have considered the optimality of such discrimination in auctions with costly entry. They showed that discrimination is suboptimal if costly entry precedes buyers' learning their values. However, "incumbent" bidders who do not face entry costs should be handicapped.

with small budget asymmetry. But a qualitatively different mechanism - "budget-handicap auction"- is optimal when budget differences are large.

Maskin (2000) studies constrained-efficient mechanisms for two and three bidders who have equal and publicly known budgets, and whose values are distributed asymmetrically. He assumes a common valuation threshold at which each bidder's budget constraint becomes binding. Yet, our analysis shows that this property does not hold generally.

Malakhov and Vohra (2008) derive an optimal dominant strategy mechanism for two buyers with discrete values, only of whom faces a limited budget.

Pai and Vohra (2014) study optimal mechanisms under private budgets and identically distributed values. In their work, the budgets and values have a finite support, with a continuous distribution considered in an extension. They provide a significant contribution to multidimensional mechanism design showing how one can work directly with reduced form auctions. In their optimal mechanism some buyer types receive separating allocations and some buyer types are pooled, although it is hard to pin down those intervals exactly. An extension of their paper considers bidders with equal and public budgets.

Although our setting with publicly known budgets is different from the one with privately known budgets in Pai and Vohra (2014), it is nevertheless interesting to compare the differential treatment of richer and poorer bidders in these two settings, since most other works focus on bidders with equal budgets. Pai and Vohra (2014) establish that "pooling serves to allot the good to disadvantaged buyer types ... even in profiles where there are buyers with higher valuations and budgets present." In contrast, in our setting handicapping of high-budget bidder occurs when budget differences are large in the region of separating allocations at low values, while the region of pooling includes high-value bidders, and in this region richer bidders get the good with a higher probability.

Che and Gale (1998) show that the first-price auction yields higher expected social surplus and revenue than the second-price auction under privately known budgets and values. Che and Gale (1996) show that the all-pay auction performs better than the first-price auction under common values and private budgets. Che and Gale (2000) explore optimal nonlinear pricing for a buyer with privately known value and budget. Zheng (2001) studies the first-price auction when buyers can bid above their budgets. In case of a win, such buyer can either use costly financing to cover the deficit or default and lose her budget. Bobkova (2020) studies first-price auction for buyers with budgets drawn from asymmetric distributions and different and commonly known values. She identifies unique equilibrium bidding distributions and shows that second-price auction provides significantly more revenue than first-price auction.

Hafalir, Ravi and Sayedi (2012) focus on a Vickrey auction for bidders with different and essentially known budgets. Their mechanism is not optimal, but is “close” to a Pareto efficient one. Borgs et. al (2005) and Dobzinski, Lavi and Nisan (2012) deal with dominant strategy mechanisms for allocating multiple goods. Both papers establish impossibility results under private budgets, the latter- for Pareto optimal allocation, the former- for allocations satisfying other properties that might be desirable. Dobzinski, Lavi and Nisan (2012) demonstrate that with public budgets, a Pareto optimal allocation can be attained by using Ausubel’s clinching auction. Baisa (2015) demonstrates that a clinching auction is a Pareto efficient mechanism under private budget constraints.

Importantly, Pareto optimality is inconsistent with the goal of revenue maximization pursued in this paper. In particular, handicapping a richer bidder, as in the budget-handicap auction, and allocating the good randomly between the bidders with values above the common threshold, as in the top auction, cannot occur in a Pareto optimal mechanism.

Che, Gale and Kim (2013a) and (2013b) and Richter (2019) study welfare-maximizing assignment of a divisible good to a continuum of budget-constrained agents. The nature of the problem studied by these authors is very different from that of our problem. In particular, as discussed in Richter (2019), his model can be reinterpreted as a single-agent problem in which budget and supply must be balanced on average, and transfers between types of this single agent are permitted. Along with considering different rationing schemes, Che, Gale and Kim (2013a) derive a surplus maximizing mechanism for buyers with private discrete values and budgets, and show that it is optimal to subsidize some buyers so that some of them pay a negative price. In our extension dealing with constrained-efficient mechanism we rule out such subsidies. Thus, their and our approaches can be seen as complementary.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 contains the main steps in the analysis. Section 4 provides the characterization of the optimal mechanism for two bidders and discusses its generalization to the case of arbitrary n bidders. Section 5 deals with constrained-efficient mechanisms. Section 6 concludes. All proofs are in the Appendix. Online Appendix available at http://www.severinov.com/bcappendixB.pdf contains extensions and numerical results.

2 Model and Preliminaries

A seller with one unit of the good faces n bidders. Bidder $i \in \{1, ..., n\}$ has a privately known value x_i for the good drawn from a commonly known probability distribution $F(.)$, which possesses a continuous positive density $f(.)$. Without loss of generality, we assume that the support of $F(.)$ is $[0,1]$. We will impose the following standard assumption on the distribution $F(.)$:[2]

Assumption 1 *Increasing Hazard rate:*

$$\frac{f(x)}{1-F(x)} \text{ is increasing in } \ x \text{ for all } x \in [0,1]$$

Bidder i with value x_i has payoff $x_i q_i - t_i$ if she gets the good with probability q_i and pays t_i. She is endowed with budget m_i which t_i can never exceed. Without loss of generality, we assume that $m_1 \geq m_2 \geq ... \geq m_n$.

The seller has zero value for the good, and her payoff is the sum of all payments $\sum_{i=1,..,n} t_i$.

By the Revelation principle (Myerson 1979) we can restrict attention to direct truthful mechanisms $Q(.) = (Q_1(.), ..., Q_n(.))$, $T(.) = (T_1(.), ..., T_n(.))$, where $Q_i(\hat{x}_1, ..., \hat{x}_n)$ is the probability that the bidder i gets the good and $T_i(\hat{x}_1, ..., \hat{x}_n)$ is the transfer that she pays to the seller when the profile of types $(\hat{x}_1, ..., \hat{x}_n)$ is announced by the bidders.

Let $q_i(x_i) = \int_{x_{-i} \in [0,1]^{n-1}} Q_i(x_i, x_{-i}) \prod_{j \neq i} dF(x_j)$ and $t_i(x_i) = \int_{x_{-i} \in [0,1]^{n-1}} T_i(x_i, x_{-i}) \prod_{j \neq i} dF(x_j)$ be the expected probability that bidder i gets the good and her expected payment, respectively, when she announces type x_i and all other bidders announce their types truthfully.

An optimal mechanism solves the expected revenue maximization problem of the seller:

$$\max \sum_{i=1,...,n} \int_{(x_1,...,x_n) \in [0,1]^n} T_i(x_1, ..., x_n) \prod_{i=1,...,n} dF(x_i) \tag{1}$$

subject to the following:

(i) *interim incentive constraints:*

$$x_i q_i(x_i) - t_i(x_i) \geq x_i q_i(\hat{x}_i) - t_i(\hat{x}_i), \quad \text{for all } (x_i, \hat{x}_i) \in [0,1]^2 \ \text{ and all } \ i \in \{1, ..., n\}. \tag{2}$$

(ii) *individual rationality constraints:*

$$x_i q_i(x_i) - t_i(x_i) \geq 0 \quad \text{for all } i \quad \text{and} \ \ x_i \in [0,1]. \tag{3}$$

(iii) *budget constraints:*

$$T_i(x_i, x_{-i}) \leq m_i \quad \text{for all } i, \ x_i \in [0,1], \ x_{-i} \in [0,1]^{n-1}. \tag{4}$$

[2] Pai and Vohra (2014) suggest that an additional assumption that $f(x_i)$ is nonincreasing is necessary for bidder i's "adjusted" virtual value to be increasing in x_i in the setting with budget constraints, which ensures that the optimal mechanism can be derived without resorting to 'ironing' techniques. However, as we show below (Lemma 6), Assumption 1 is sufficient to guarantee the monotonicity of the virtual values in the optimal mechanism with budget constraints, and this extra assumption on $f(.)$ is unnecessary.

(iv) *feasibility constraints*:

$$\sum_{i=1}^{n} Q_i(x_1, ..., x_n) \leq 1 \text{ and } Q_i(x_1, ..., x_n) \geq 0 \text{ for all } i \text{ and } (x_1, ..., x_n) \in [0,1]^n. \tag{5}$$

3 Analysis

Our first result establishes the existence of an optimal mechanism.

Lemma 1 *There exists an optimal mechanism* $(Q_1(.), ..., Q_n(.), T_1(.), ..., T_n(.))$ *solving the problem (1) subject to (2)-(5).*

Proof of Lemma 1: The maximization objective in (1) is a continuous linear functional in the Hilbert space $L^2([0,1]^n)$ with convex admissible set specified by (2)-(5). Therefore, the solution to this problem exists (see e.g. Theorem 2.6.1 in Balakrishnan (1993)). *Q.E.D.*

Next, let $U_i(x_i) \equiv q_i(x_i)x_i - t_i(x_i)$ be the net expected payoff of buyer i of type x_i in the mechanism under truthtelling. The following result is standard and is left without proof:[3]

Lemma 2 *A mechanism* $(Q_1(.), ..., Q_n(.), T_1(.), ..., T_n(.))$ *is incentive compatible and individually rational if and only if* $q_i(x_i)$ *is increasing in* x_i *for all* i *and* $x_i \in [0,1]$*, and:*

$$U_i(x_i) = \int_0^{x_i} q_i(s)ds + c_i \text{ for some } c_i \in \mathbb{R}_+$$

The individual rationality requires that $c_i \geq 0$. The optimality then implies that $c_i = 0$. Given this, we drop c_i altogether from the analysis. So, from Lemma 2 it follows that:

$$t_i(x_i) = x_i q_i(x_i) - \int_0^{x_i} q_i(s)ds \tag{6}$$

Consider now the budget constraints. First, we can replace the ex-post budget constraint in (4) with the interim one, $t_i(x_i) \leq m_i$ for all i and x_i. Indeed, the interim budget constraints obviously hold when (4) holds. In the opposite directions, if $t_i(x_i) \leq m_i$ for all i and x_i, then (4) holds if we set $T_i(x_i, x_{-i}) = t_i(x_i)$ for all i, x_i and x_{-i}. Doing so does not affect the seller's objective, the incentive or individual rationality constraints, since these depend only on the expected transfers $t_i(.)$, but it can potentially relax the budget constraint under some type profiles since the maximal payment by bidder i weakly decreases.

Next, for a mechanism $(q_i(.), t_i(.))$ let us define the threshold $\bar{x}_i$ as follows:

$$\bar{x}_i = \inf\{x_i \in [0,1] | t_i(x_i) = t_i(1)\} \tag{7}$$

[3] Since the budgets are common knowledge and do not vary with bidders' values, the budgets cannot be used as a screening device: any transfer that is feasible for one type is feasible for another type of a bidder.

Lemma 3 *Suppose that* $(q_i(.), t_i(.))$ *is an incentive compatible individually rational mechanism. If* $\bar{x}_i < 1$, *then* $t_i(x_i)$ *and* $q_i(x_i)$ *are constant on the interval* $(\bar{x}_i, 1]$. *Moreover, without loss of generality in an optimal mechanism we can take* $t_i(\bar{x}_i) = t_i(1)$ *and* $q_i(\bar{x}_i) = q_i(1)$.

To understand the last statement of Lemma 3 note that, as we show below, in the optimal mechanism $q_i(.)$ and hence $t_i(.)$ are discontinuous at $\bar{x}_i$ for at least some i (see Figure 2), and then both left- and right-hand limits of the allocations at $\bar{x}_i$ are incentive compatible and individually rational for type $\bar{x}_i$. Which of these two limit allocations is assigned to $\bar{x}_i$ does not affect the seller's profits. However, it is more convenient for notational purposes to choose the right limit and set $t_i(\bar{x}_i) = t_i(1)$ and $q_i(\bar{x}_i) = q_i(1)$.

The threshold values $(\bar{x}_1, ..., \bar{x}_n)$ play an important role in our analysis as the key choice variables which ultimately determine the whole mechanism. Lemma 3 and equation (6) imply that budget constraints $t_i(x_i) \leq m_i$, $x_i \in [0, 1]$ can be replaced with the inequality:

$$m_i \geq \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s)ds, \quad i \in \{1, ..., n\} \tag{8}$$

The budget constraint of bidder i is binding (non-binding, or never binding) when (8) holds as an equality (strict inequality). Below we will provide conditions on budgets under which all bidders' budget constraints are binding, and under which only some bidders have binding budget constraints in the optimal mechanism. Our analysis applies to both cases.

Next, replacing the transfers in the objective (1) with the right-hand side of (6), using $q_i(x_i) = q_i(\bar{x}_i)$ for all $x_i \in [\bar{x}_i, 1]$, and then integrating by parts yields the modified objective:

$$\begin{aligned}\sum_{i=1}^{n} \int_0^1 t_i(x_i)dF(x_i) &= \sum_{i=1}^{n} \int_0^1 \left(q_i(x_i)x_i - \int_0^{x_i} q_i(x)dx \right) dF(x_i) \\ &= \sum_{i=1}^{n} \int_0^{\bar{x}_i} q_i(x_i) \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) dF(x_i) + \sum_{i=1}^{n} \int_{\bar{x}_i}^1 q_i(\bar{x}_i)\bar{x}_i dF(x_i),\end{aligned} \tag{9}$$

which has to be maximized subject to the feasibility constraints (5), budget constraints (8), and the constraint that $q_i(.)$ is increasing for all i. We refer to this as full unrelaxed problem.

Following standard approach we will solve the relaxed problem maximizing (9) subject to (5) and (8) and omitting the monotonicity constraint on $q_i(.)$. We will then show that the solution to the relaxed problem is such that $q_i(.)$ is increasing, strictly at $\bar{x}_i$ from the left. The latter ensures that (7) holds, i.e., $\bar{x}_i$ is the lowest type of i making the largest transfer.

The relaxed problem has the following Lagrangian in maximizing which we take care of the feasibility constraints (5) by imposing them directly on the trading probabilities Q:

$L(Q, \bar{x}, \lambda) =$

$$\sum_{i=1}^{n}\left(\int_{0}^{\bar{x}_i} q_i(x_i)\left(x_i - \frac{1-F(x_i)}{f(x_i)}\right)dF(x_i) + \int_{\bar{x}_i}^{1} q_i(\bar{x}_i)\bar{x}_i dF(x_i) - \lambda_i\left(q_i(\bar{x}_i)\bar{x}_i - \int_{0}^{\bar{x}_i} q_i(x)dx - m_i\right)\right)$$

$$= \sum_{i=1}^{n}\left(\int_{0}^{\bar{x}_i} q_i(x_i)\left(x_i - \frac{1-\lambda_i-F(x_i)}{f(x_i)}\right)dF(x_i) + \int_{\bar{x}_i}^{1} q_i(\bar{x}_i)\left(\bar{x}_i - \frac{\lambda_i\bar{x}_i}{1-F(\bar{x}_i)}\right)dF(x_i) + \lambda_i m_i\right) \tag{10}$$

where $\lambda_i \geq 0$ is a Lagrange multiplier associated with bidder i's budget constraint (8).

Using $q_i(x_i) = \int_{x_{-i}\in[0,1]^{n-1}} Q_i(x_i, x_{-i}) \prod_{j\neq i} dF(x_j)$ and changing the order of summation and integration in (10) we can rewrite it as follows:

$$L(Q, \bar{x}, \lambda) = \int_{(x_1,...,x_n)\in[0,1]^n} \sum_{i=1}^{n} Q_i(x_1, ..., x_n)\gamma_i(x_i) \prod_{i=1,...,n} dF(x_i) + \sum_{i=1}^{n} \lambda_i m_i. \tag{11}$$

where $\gamma_i(x_i)$ is defined as follows for $i \in \{1, ..., n\}$:

$$\gamma_i(x_i) = \begin{cases} x_i - \frac{1-\lambda_i-F(x_i)}{f(x_i)}, & \text{if} \quad x_i < \bar{x}_i, \\ \bar{x}_i\left(1 - \frac{\lambda_i}{1-F(\bar{x}_i)}\right), & \text{if} \quad x_i \geq \bar{x}_i. \end{cases} \tag{12}$$

As one can see from (11), $\gamma_i(.)$ plays the role of the virtual value of bidder i. Recall that without budget constraints, i's virtual value is $x_i - \frac{1-F(x_i)}{f(x_i)}$. So budget constraints cause the virtual value of type $x_i \in [0, \bar{x}_i)$ to increase by an amount proportional to the value of the Lagrange multiplier. Intuitively, this happens because when $\lambda_i > 0$, then all types above $\bar{x}_i$ pay their whole budget. So the seller cannot extract more surplus from these types, and increasing the probability with which the types in $[0, \bar{x}_i)$ get the good depresses the seller's profits by less than without budget constraints.

On the other hand, all types in the endogenous "atom" $[\bar{x}_i, 1]$ get the same allocation, and have the same virtual value, $\bar{x}_i\left(1 - \frac{\lambda_i}{1-F(\bar{x}_i)}\right)$. The latter is decreasing in λ_i as a higher λ_i is associated with a smaller budget which decreases the seller's profits. The ratio $\frac{\lambda_i}{1-F(\bar{x}_i)}$ gives the average (per type) value of this effect on the types in $[\bar{x}_i, 1]$. Furthermore, $\bar{x}_i\left(1 - \frac{\lambda_i}{1-F(\bar{x}_i)}\right)$ is nonnegative in the optimal mechanism because, as shown in Lemma 5, $\lambda_i \leq 1 - F(\bar{x}_i)$.

Inspection of (11) yields the following Lemma:

Lemma 4 *Any solution to the relaxed problem (11) is such that for bidder i and $(x_i, x_{-i}) \in [0, 1]^n$:*

1. $Q_i(x_i, x_{-i}) = 1$ *if* $\gamma_i(x_i) > \max\{0, \max_{j\neq i}\gamma_j(x_j)\}$*;*

2. $Q_i(x_i, x_{-i}) = 0$ *if* $\gamma_i(x_i) < \max\{0, \max_{j\neq i}\gamma_j(x_j)\}$*.*

3. $\sum_{i=1}^n Q_i(x_1, ..., x_n) = 1$ *if* $\max_i \gamma_i(x_i) > 0$*.*

According to Lemma 4, the profile of virtual values $(\gamma_1(x_1), ..., \gamma_n(x_n))$ determines the trading probabilities $(Q_i(x), ..., Q_n(x))$ uniquely except in case of a tie, when two or more bidders have the highest virtual value. The ties that have zero probability can be ignored, as the seller can resolve them arbitrarily without affecting her profits. In particular, this applies to ties that involve a type $x_i \in [0, \bar{x}_i)$. However, if $\gamma_i(\bar{x}_i) = \gamma_j(\bar{x}_j)$ for some $j \neq i$, then all types in $[\bar{x}_i, 1]$ are tied with all types in $[\bar{x}_j, 1]$. This tie has a positive probability. As we show below, such ties occur if the bidders' budgets are sufficiently close to each other, and the tie-breaking rule in this case is determined by their binding budget constraints (8).

Lemma 4 implies that at the optimum $\sum_{i=1}^n Q_i(x_1, ..., x_n)\gamma_i(x_i) = \max\{0, \max_i \gamma_i(x_i)\}$ for all $x = (x_1, ..., x_n) \in [0, 1]^n$. Using this in (11) allows us to replace Lagrangian (11) with the following objective that depends only on $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$ and $\lambda = (\lambda_1, ..., \lambda_n)$:

$$\mathcal{L}(\bar{x}, \lambda) = \max_{Q:\, 0\leq Q_i(x)\leq 1,\, \sum_i Q_i(x)\leq 1} L(Q, \bar{x}, \lambda) = \int_{x\in[0,1]^n} \max\{0, \max_{i=1,...,n} \gamma_i(x_i)\} \prod_i dF(x_i) + \sum_{i=1}^n \lambda_i m_i. \tag{13}$$

Thus, solving the relaxed problem (11) boils down to finding the profile $(\bar{x}, \lambda)$ solving the following (primal) problem:

$$\max_{\bar{x}\in[0,1]^n} \min_{\lambda\in\mathbf{R}_+^n} \mathcal{L}(\bar{x}, \lambda), \tag{14}$$

from which we then recover the probabilities of trading using Lemma 4 and the budget constraints (8). The next Lemma provides an important step towards solving this problem by establishing an optimal relationship between $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$ and $\lambda = (\lambda_1, ..., \lambda_n)$. Define:

$$\gamma_i^-(\bar{x}_i) \equiv \lim_{x_i\uparrow\bar{x}_i} \gamma_i(x_i) = \bar{x}_i - \frac{1 - \lambda_i - F(\bar{x}_i)}{f(\bar{x}_i)}. \tag{15}$$

Then we have:

Lemma 5 *In any solution to the relaxed program (11), the profile $(\bar{x}, \lambda)$ is such that:*

1. *For all i s.t. $\bar{x}_i \leq \bar{x}_j$ for some $j \neq i$, the virtual values satisfy $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$, or, equivalently,*

$$\lambda_i = \frac{(1 - F(\bar{x}_i))^2}{1 - F(\bar{x}_i) + \bar{x}_i f(\bar{x}_i)}. \tag{16}$$

2. *If there exists bidder h_1 such that $\bar{x}_{h_1} > \max_{j \neq h_1} \bar{x}_j$, then $\bar{x}_{h_1} < 1$ and the virtual values satisfy $\gamma_{h_1}(\bar{x}_{h_1}) > \gamma^-_{h_1}(\bar{x}_{h_1}) = \max_{j \neq h_1} \gamma_j(\bar{x}_j)$ or, equivalently,*
$$\bar{x}_{h_1} - \frac{1 - F(\bar{x}_{h_1}) - \lambda_{h_1}}{f(\bar{x}_{h_1})} = \max_{j \neq h_1} \frac{\bar{x}_j^2 f(\bar{x}_j)}{1 - F(\bar{x}_j) + \bar{x}_j f(\bar{x}_j)}. \tag{17}$$

The proof of Lemma 5 relies on the following observation. If some bidder i's virtual value is not continuous at $\bar{x}_i$ and the maximum of the other bidders' virtual values belongs to the wedge between the left and right limits of $\gamma_i(\bar{x}_i)$ with a positive probability, then the seller can attain a higher payoff by raising $\bar{x}_i$. The intuition behind this is as follows. Raising $\bar{x}_i$ has two effects. First, it changes the virtual value $\gamma_i(\bar{x}_i) = \bar{x}_i \left(1 - \frac{\lambda_i}{1-F(\bar{x}_i)}\right)$. Second, it reduces the interval $[\bar{x}_i, 1]$ on which i's virtual value is $\gamma_i(\bar{x}_i)$, rather than $\gamma_i(x_i) = x_i - \frac{1-F(x_i)-\lambda_i}{f(x_i)}$. These two effects shift the seller's profits in different directions. If $\gamma_i(\bar{x}_i) > \gamma^-_i(\bar{x}_i)$, then raising $\bar{x}_i$: (i) increases $\gamma_i(\bar{x}_i)$, which raises the profits; (ii) reduces the interval on which $\gamma_i(\bar{x}_i)$ applies, which lowers the profits. If $\gamma_i(\bar{x}_i) < \gamma^-_i(\bar{x}_i)$, these two effects have the opposite influences on profits. When the probability that bidder i gets the good is the same whether her virtual value is $\gamma_i(\bar{x}_i)$ or $\gamma^-_i(\bar{x}_i)$, these two effects have the same magnitude and balance each other. However, if the maximum of the other bidders' virtual values belongs to the wedge between the left and right limits of $\gamma_i(\bar{x}_i)$ with a positive probability, then bidder i is awarded the good with a higher probability when her virtual value is the higher of $\gamma_i(\bar{x}_i)$ and $\gamma^-_i(\bar{x}_i)$ than when it is the lower of the two. Therefore, the negative effect on profits of raising $\bar{x}_i$ has a smaller magnitude than the positive effect, and so it is optimal to raise $\bar{x}_i$.

If $\bar{x}_i \leq \bar{x}_j$ for some j, then $\gamma_i(.)$ must be continuous at $\bar{x}_i$, because otherwise the maximum of the other bidders' virtual values would belong to the wedge between $\gamma_i(\bar{x}_i)$ and $\gamma^-_i(\bar{x}_i)$. However, this is not the case for bidder h_1 with the strictly highest threshold: we can have $\gamma_{h_1}(\bar{x}_{h_1}) > \gamma^-_{h_1}(\bar{x}_{h_1})$ because, as long as $\gamma^-_{h_1}(\bar{x}_{h_1}) \geq \max_{j \neq h_1} \gamma_j(\bar{x}_j)$, no bidder's virtual value falls in the wedge $[\gamma^-_{h_1}(\bar{x}_{h_1}), \gamma_{h_1}(\bar{x}_{h_1})]$. Yet $\gamma^-_{h_1}(\bar{x}_{h_1})$ is pinned down by $\gamma^-_{h_1}(\bar{x}_{h_1}) = \max_{j \neq h_1} \gamma_j(\bar{x}_j)$ (equation (17)), which follows from the definition of $\bar{x}_{h_1}$ as the lowest type of h_1 paying the highest transfer. This equation and the fact that $\bar{x}_{h_1}$ is the strictly highest threshold imply that $\lambda_{h_1} < \frac{\left(1-F\left(\bar{x}_{h_1}\right)\right)^2}{1-F\left(\bar{x}_{h_1}\right)+\bar{x}_{h_1} f\left(\bar{x}_{h_1}\right)}$ or, equivalently, $\gamma_{h_1}(\bar{x}_{h_1}) > \gamma^-_{h_1}(\bar{x}_{h_1})$.

Lemma 5 has a number of important implications that we present in the rest of this section. First, (16) and (17) define a bijection between the set of thresholds $(\bar{x}_1, ..., \bar{x}_n)$ and the set of multipliers $(\lambda_1, ..., \lambda_n)$, which we confirm formally in Corollary 1 in the Appendix below the proof of Lemma 5. This allows us to reduce the number of choice variables from $2n$ to n, which provides an important step towards solving problem (14) and deriving the optimal mechanism.

Next, observe from (16) and (17) that in any solution to problem (14) we have $\lambda_i \leq 1 - F(\bar{x}_i)$ for all i. This property is behind the following result:

Lemma 6 *Any solution to the relaxed problem (11) is such that $\gamma_i(.)$ is strictly increasing and $q_i(.)$ is increasing on $[0, \bar{x}_i]$ for all i. So, this solution also solves the full unrelaxed problem maximizing (9) subject to (5), (8), and the constraint that $q_i(.)$ is increasing.*

Combining Lemmas 4 and 6 we can now provide an explicit expression for $q_i(x_i)$.

Lemma 7 *In an optimal mechanism, the expected probability of trading $q_i(x_i)$ satisfies:*

$$Prob.\left[\gamma_i(x_i) > \max\{0, \max_{j \neq i} \gamma_j(x_j)\} \,|x_i\right] \leq q_i(x_i) \leq Prob.\left[\gamma_i(x_i) \geq \max\{0, \max_{j \neq i} \gamma_j(x_j)\} \,|x_i\right] \quad (18)$$

The inequalities in (18) hold as equalities for almost all $x_i \in [0, \bar{x}_i)$ and for $\bar{x}_i$ if $\bar{x}_i \neq \bar{x}_j$ for all $j \neq i$. So the profile $(\bar{x}_1, ..., \bar{x}_n)$ (equivalently, profile $(\lambda_1, ..., \lambda_n)$) uniquely determines $q_i(x_i)$ for almost all $x_i \in [0, \bar{x}_i)$, and $q_i(\bar{x}_i)$ if $\bar{x}_i \neq \bar{x}_j$ for all j, $j \neq i$.

We can now establish the following intuitive relationship between budgets and thresholds:

Lemma 8 *If $m_i > m_j$ for some $i, j \in \{1, ..., n\}$, then in an optimal mechanism $\bar{x}_i \geq \bar{x}_j$.*

Next, we show that a bidder's budget constraint is binding in an optimal mechanism if her budget is below the optimal price for a seller facing a single bidder. This result is noteworthy because the budget level at which a bidder's budget constraint is binding is endogenous to the mechanism and, in an optimal mechanism, depends on the other bidders' budgets.

Lemma 9 *If $m_i \leq \arg\max_p p(1 - F(p))$, then the budget constraint of bidder i is binding in the optimal mechanism, i.e., (8) holds as equality.*

To complete the derivation of the optimal mechanism we will use the duality theory. Recall that by Lemma 5, for every $\bar{x} \in [0,1]^n$ any $\lambda \in \mathbf{R}^n_+$ which minimizes $\mathcal{L}(\bar{x}, \lambda)$ is such that $\lambda_i \leq 1$ for all i. Therefore, our primal problem in (14) is equivalent to the problem $\max_{\bar{x} \in [0,1]^n} \min_{\lambda \in [0,1]^n} \mathcal{L}(\bar{x}, \lambda)$. The dual to this problem is $\min_{\lambda \in [0,1]^n} \max_{\bar{x} \in [0,1]^n} \mathcal{L}(\bar{x}, \lambda)$. Solving the dual problem involves minimizing the following dual function $g(\lambda)$ on $[0,1]^n$:

$$g(\lambda) \equiv \mathcal{L}(\bar{x}(\lambda), \lambda) = \max_{\bar{x}} \mathcal{L}(\bar{x}, \lambda) = \max_{\bar{x}} \int_{x \in [0,1]^n} \max\{0, \max_{i=1,...,n} \gamma_i(x_i)\} \prod_i dF(x_i) + \sum_{i=1}^{n} \lambda_i m_i, \quad (19)$$

where $\bar{x}(\lambda)$ is the optimal threshold profile under a given profile of multipliers λ defined by equations (16) and (17) in Lemma 5.

Importantly, the next Lemma establishes that our primal problem possesses strong duality property and both problems have a unique solution:

Lemma 10 *The primal problem has strong duality property, i.e.,*

$$\max_{\bar{x}\in[0,1]^n} \min_{\lambda\in[0,1]^n} \mathcal{L}(\bar{x},\lambda) = \min_{\lambda\in[0,1]^n} \max_{\bar{x}\in[0,1]^n} \mathcal{L}(\bar{x},\lambda)$$

The solution to the dual and primal problems is unique.

To prove the strong duality we establish that $\mathcal{L}(\bar{x},\lambda)$ possesses saddle-point property. The uniqueness of the solution is shown by establishing that the dual function $g(\lambda)$ is strongly convex. Its unique minimizer determines a unique vector of optimal thresholds $\bar{x}$ according to Lemma 5. So, we can solve problem (14) and derive the optimal mechanism by minimizing the dual function $g(\lambda)$.

4 Main Results

In this section we provide the results characterizing the optimal mechanism. For ease of exposition, we focus on the case of two bidders. The generalization of these results to n bidders is described in subsection 4.1 and the details are provided in the online Appendix.

Recall that without loss of generality we take that $m_1 \geq m_2$. Additionally, in the sequel we will assume that $m_2 < \widehat{m} \equiv 1 - \int_{r:r-\frac{1-F(r)}{f(r)}=0}^{1} F(x)dx$. For, if $m_2 \geq \widehat{m}$, then the optimal mechanism is the standard optimal auction for the environment without budget constraints, which is feasible since in this auction the highest bidder's transfer is $\widehat{m}$. On the other hand, if $m_2 < \widehat{m}$, then the standard auction is infeasible and, as will show below, in the optimal mechanism bidder 2's budget constraint is binding for types $[\bar{x}_2, 1]$ for some $\bar{x}_2 < 1$.

The optimal mechanism also depends on bidder 1's budget. If the difference $m_1 - m_2$ is sufficiently small, then in the optimal mechanism the bidders face the same threshold $\bar{x}^t < 1$ above which their budget constraints are binding. Because the bidders have unequal budgets, the seller discriminates between them "at the top:" a richer bidder with value in $[\bar{x}^t, 1]$ gets the good with a higher probability and pays more (her budget) than a poorer bidder with the value in this range. In contrast, both bidders with values below $\bar{x}^t$ are treated symmetrically, and the one with the higher value (above the reservation level) gets the good. We refer to this mechanism as "top auction." But when $m_1 - m_2$ is sufficiently large, a qualitatively different mechanism -"budget-handicap auction"- is optimal.

The next Theorem characterizes the "top auction" and the conditions for its optimality.

Theorem 1 *Suppose that $m_2 < \widehat{m}$. Then in the optimal mechanism the bidders have a common threshold $\bar{x}^t < 1$ and the budget constraint of bidder $i \in \{1, 2\}$ is binding for any*

$x_i \in [\bar{x}^t, 1]$, *if and only if the following two conditions hold:*

$$m_1 + m_2 = \bar{x}^t(1 + F(\bar{x}^t)) - 2\int_{r_t}^{\bar{x}^t} F(x)dx, \tag{20}$$

$$m_1 - m_2 \le \bar{x}^t(1 - F(\bar{x}^t)). \tag{21}$$

where r_t *is the reservation value uniquely defined by* $r_t = \frac{1-F(r_t)-\frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r_t)}$.

In this mechanism, referred to as "top auction," $q_i(x_i) = F(x_i)$ *for all* $x_i \in [r_t, \bar{x}^t)$, $q_i(x_i) = 0$ *for all* $x_i \in [0, r_t)$, *and* $q_i(\bar{x}^t)$ *is uniquely defined by* i*'s binding budget constraint:*

$$m_i = \bar{x}^t q_i(\bar{x}^t) - \int_{r_t}^{\bar{x}^t} F(s)ds, \qquad i \in \{1, 2\}. \tag{22}$$

Note that (20) determines $\bar{x}^t$ uniquely, with $\bar{x}^t$ increasing in $m_1 + m_2$, when $m_1 \le \widehat{m} \equiv 1 - \int_{r:r-\frac{1-F(r)}{f(r)}=0}^{1} F(x)dx$.[4] When $m_1 > \widehat{m}$, (20) and (21) cannot hold.[5]

For bidder i with value in $[r_t, \bar{x}^t)$ the top auction looks exactly like a standard symmetric all-pay auction: she gets the good when her competitor has a lower value, and so $q_i(x_i) = F(x_i)$. The reservation value r_t is below the reservation value in the optimal auction without budget constraints r, which satisfies $r - \frac{1-F(r)}{f(r)} = 0$. This is so because with budget constraints the tradeoff between higher efficiency at lower values and leaving greater surplus to the bidders with higher values shifts towards the former, since the bidders with values above their thresholds pay the whole budgets and no more surplus can be extracted from them.

At the threshold $\bar{x}^t$ both budget constraints become binding, so bidder i with value in $[\bar{x}^t, 1]$ pays her whole budget. Naturally, $q_1(\bar{x}^t) \ge q_2(\bar{x}^t)$ because $m_1 \ge m_2$. So the top auction discriminates only between bidders with high values. In fact, both $q_1(x)$ and $q_2(x)$ jump at $x = \bar{x}^t$, except in the borderline parameter case in which only $q_1(x)$ jumps (to 1). Randomization between the bidders at the top generates an inefficiency compared to the standard optimal auction: a bidder with a lower value in $[\bar{x}^t, 1]$ may end up getting the good even if the other bidder has a higher value. Yet, in a sense, the top auction introduces a minimal amount of additional inefficiency compared to the standard optimal auction. Therefore, under a fixed aggregate budget $m_1 + m_2$ the seller gets more revenue

[4] To confirm this, note that the right-hand side of (20) is: (i) zero when $\bar{x}^t = 0$; (ii) exceeds $m_1 + m_2$ when $\bar{x}^t = 1$ given that $m_1 \le \widehat{m}$; (iii) is increasing in $\bar{x}^t$. The latter is true because the derivative of the right-hand side w.r.t $\bar{x}^t$ is equal to $1 - F(\bar{x}^t) + \bar{x}^t f(\bar{x}^t) + 2F(r(\bar{x}^t))\frac{dr_t}{d\bar{x}^t}$, which is positive, in particular, because $\frac{dr_t}{d\bar{x}^t} > 0$.

[5] To see this, add (20) and (21) to obtain $m_1 \le \bar{x}^t - \int_{r_t}^{\bar{x}^t} F(x)dx$ and note that $\bar{x}^t - \int_{r_t}^{\bar{x}^t} F(x)dx \le 1 - \int_{r:r-\frac{1-F(r)}{f(r)}=0}^{1} F(x)dx = \widehat{m}$ for all $\bar{x}^t \in [0, 1]$.

from a top auction than from the other mechanism, "budget-handicap" auction, which is optimal when $m_1 - m_2$ is sufficiently large (Theorem 3).

The top auction can be implemented via an indirect mechanism in which a bidder is offered a choice between an all-pay auction and a lottery. If she chooses the lottery, bidder i pays m_i for a "lottery ticket" which wins the good with probability $q_i(\bar{x}^t)$. If she chooses the all-pay auction, she gets the good if her bid b_i is above the "reserve price" $r_t F(r_t)$ and the other bidder also participates in the all-pay auction and bids less than b_i. It is easy to see that the optimal strategy of bidder i is to buy the lottery ticket if $x_i \in [\bar{x}^t, 1]$; to bid $b_i(x_i) = x_i F(x_i) - \int_{r^t}^{x_i} F(s)ds$ if $x_i \in [r_t, \bar{x}^t)$; and not to participate if $x_i < r^t$.

To understand the optimality conditions (20)-(21) better, first consider (20). It is the aggregate of the bidders' binding budget constraints at the threshold $\bar{x}^t$. To obtain it, we sum $m_i = \bar{x}^t q_i(\bar{x}^t) - \int_{r_t}^{\bar{x}^t} F(s)ds$ over $i \in \{1, 2\}$ and take into account that $q_i(x_i) = F(x_i)$ for $x_i \in [r_t, \bar{x}^t)$ and $q_1(\bar{x}^t) + q_2(\bar{x}^t) = 1 + F(\bar{x}^t)$. The latter holds because by Lemma 4 the good is allocated to a bidder with valuation in $[\bar{x}^t, 1]$ whenever there is such a bidder.

Condition (21) requires $m_1 - m_2$ to be sufficiently small, for otherwise it is impossible for both bidders' budget constraints to be binding at $\bar{x}^t$. Indeed, since $q_1(\bar{x}^t) \leq 1$, we have:

$$m_1 = \bar{x}^t q_1(\bar{x}^t) - \int_{r_t}^{\bar{x}^t} F(s)ds \leq \bar{x}^t - \int_{r_t}^{\bar{x}^t} F(s)ds \tag{23}$$

On the other hand, $q_2(\bar{x}^t) \geq F(\bar{x}^t)$ since $q_2(.)$ must be increasing for incentive compatibility and $q_2(x) = F(x)$ on $[0, \bar{x}^t)$. So,

$$m_2 = \bar{x}^t q_2(\bar{x}^t) - \int_{r_t}^{\bar{x}^t} F(s)ds \geq \bar{x}^t F(\bar{x}^t) - \int_{r_t}^{\bar{x}^t} F(s)ds. \tag{24}$$

Combining the inequalities in (23) and (24) yields (21). Conversely, it is easy to see that (20) and (21) imply the inequalities in (23) and (24). Thus, (20) and (21) are necessary conditions for the bidders to have a common threshold. The surprising conclusion from Theorem 1 is that they are also sufficient for the optimality of a mechanism with a common threshold.

Now suppose that $m_1 - m_2$ is sufficiently large, so that (20) and (21) fail to hold. Then the inequalities in (23) and (24) also fail, and therefore it is impossible to construct a mechanism in which each bidder with a value above a common threshold $\bar{x}^t$ pays her budget and the necessary conditions $q_1(\bar{x}^t) \leq 1$ and $F(\bar{x}^t) \leq q_2(\bar{x}^t)$ hold.

So, the seller has to offer a different mechanism using additional tools to discriminate between the bidders. First, she sets different thresholds $\bar{x}_1$ and $\bar{x}_2$, with $\bar{x}_1 > \bar{x}_2$, and strongly favors richer bidder 1 over poorer bidder 2 at high values by setting $q_1(\bar{x}_1) = 1$ and $q_2(\bar{x}_2) = F(\bar{x}_1)$. Notably, the seller now uses another kind of price discrimination that works in the opposite direction. The poorer bidder 2 with a value below $\bar{x}_2$ has a higher probability of trading and faces a lower reserve price than a richer bidder 1 with the same value. This motivates the use of the term "budget-handicap auction" to describe this mechanism. The handicapping of bidder 1 with low values creates stronger competition for her from bidder 2 and allows the seller to extract a higher payment from bidder 1 with values above $\bar{x}_1$. But it also increases the payoff of bidder 2 and lowers the surplus generated by bidder 1 with intermediate values. As a result, the seller' expected revenue in the budget handicap auction is lower than in the top auction in the situation with the same aggregate budget, but a smaller budget difference (Theorem 3).

These qualitative properties of the "budget-handicap" auction hold irrespectively of whether both bidders' budget constraints are binding in it, which occurs when m_1 is sufficiently small, or only the budget constraint of bidder 2 is binding, which occurs when m_1 is large enough. Yet, the thresholds $\bar{x}_1$ and $\bar{x}_2$ are determined differently in these two cases. In fact, the determination of $\bar{x}_1$ and $\bar{x}_2$ is a two-step procedure. We start with the mechanism for the case when bidder 1's budget constraint is not binding, and so $\bar{x}_1$ is the lowest type of bidder 1 paying the highest transfer which is still below m_1. In this case, $\bar{x}_1$ and $\bar{x}_2$ are jointly determined by bidder 2's binding budget constraint and the optimality condition (17) in Lemma 5 with $\lambda_1 = 0$ (restated as equation (26) in Theorem 2). With $q_i(x_i)$, $i \in \{1, 2\}$, determined by $\bar{x}_1$ and $\bar{x}_2$ according to Lemma 7, the highest transfer paid by bidder 1 in this mechanism is $t_1(\bar{x}_1) = \bar{x}_1 - \int_0^{\bar{x}_1} q_1(x_1)dx_1$. So, this mechanism is optimal if $t_1(\bar{x}_1) \leq m_1$.[6]

However, if $m_1 < t_1(\bar{x}_1)$, then the constructed mechanism is not feasible, and both budget constraints must be binding, i.e., we must have: $m_1 = \bar{x}_1 - \int_0^{\bar{x}_1} q_1(x)dx$ and $m_2 = \bar{x}_2 F(\bar{x}_1) - \int_0^{\bar{x}_1} q_2(x)dx$. The thresholds $\bar{x}_1$ and $\bar{x}_2$ are then determined as the solution to these equations with $q_i(x_i)$, $x_i \in [0, \bar{x}_i)$, specified according to Lemma 7. Formally, we have:

Theorem 2 *Suppose that $m_2 < \widehat{m}$ and conditions (20)-(21) do not hold. Then in the optimal mechanism, referred to as "budget-handicap auction," the bidders have different thresholds, $\bar{x}_1$ and $\bar{x}_2$, with $\bar{x}_1 > \bar{x}_2$.*

In this mechanism, $q_1(\bar{x}_1) = 1$, $q_2(\bar{x}_2) = F(\bar{x}_1)$, $q_i(x_i) = 0$ for $x_i \in [0, r_i)$, $i \in \{1, 2\}$, $q_1(x) < F(x)$ for all $x \in [r_1, \bar{x}_1)$ and $q_2(x) > F(x)$ for all $x \in [r_2, \bar{x}_2)$, where the reservation values r_1 and r_2 satisfy $r_1 > r_2$[7].

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[6]The case where $t_1(\bar{x}_1) = m_1$ is the boundary case in which the budget constraint of bidder 1 is binding and this mechanism is optimal.

[7]Precisely, r_1 and r_2 are uniquely defined by the following equations:
$r_1 - \frac{-F(r_1)+F(\bar{x}_1)+f(\bar{x}_1)\left(\bar{x}_1 - \frac{\bar{x}_2^2 f(\bar{x}_2)}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}\right)}{f(r_1)} = 0$ and $r_2 - \frac{1-F(r_2)-\frac{(1-F(\bar{x}_2))^2}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}}{f(r_2)} = 0$

The optimal thresholds $\bar{x}_1$ and $\bar{x}_2$ are determined as follows:

(i) $\bar{x}_1$ and $\bar{x}_2$ are the unique solution to the following system of equations:

$$m_2 = \bar{x}_2 F(\bar{x}_1) - \int_0^{\bar{x}_2} q_2(x_2)dx_2, \tag{25}$$

$$\bar{x}_1 - \frac{1 - F(\bar{x}_1)}{f(\bar{x}_1)} = \frac{\bar{x}_2^2 f(\bar{x}_2)}{1 - F(\bar{x}_2) + \bar{x}_2 f(\bar{x}_2)}, \tag{26}$$

provided that this solution is such that bidder 1's budget constraint it satisfied, i.e.,

$$m_1 \geq \bar{x}_1 - \int_0^{\bar{x}_1} q_1(x_1)dx_1. \tag{27}$$

(ii) If the solution to (25) and (26) does not satisfy (27), then $\bar{x}_1$ and $\bar{x}_2$ are the unique solution to the system of binding budget constraints that consists of (25) and:

$$m_1 = \bar{x}_1 - \int_0^{\bar{x}_1} q_1(x_1)dx_1. \tag{28}$$

Note that all elements of the mechanism in part (i) of Theorem 2, including $\bar{x}_1$ and $q_1(.)$, are determined by m_2 through the budget constraint (25). So letting $\tilde{m}(m_2)$ denote the right-hand side of (27), the Theorem shows that the budget constraint of bidder 1 is not binding in the optimal mechanism if and only if $m_1 > \tilde{m}(m_2)$. As established in the proof of the Theorem, $\tilde{m}(m_2) > m_2$ for all $m_2 < \widehat{m}$. In the boundary case where (27) holds as equality, i.e., $m_1 = \tilde{m}(m_2)$, the solutions from parts (i) and (ii) of Theorem 2 coincide.

As also shown in the proof, $\lim_{x_i \to \bar{x}_i - 0} q_i(x_i) = F(\bar{x}_j)$ for $i, j \in \{1, 2\}$. Since $q_1(\bar{x}_1) = 1$ and $q_2(\bar{x}_2) = F(\bar{x}_1)$, it follows that $q_1(.)$ jumps from $F(\bar{x}_2)$ to 1 at $\bar{x}_1$, while $q_2(.)$ continuously reaches its maximum $F(\bar{x}_1)$ at $\bar{x}_2$, as illustrated in Figure 1b.

The implementation of the budget-handicap auction via an indirect mechanism is similar to that for the top-auction, with bidder i offered a choice between bidding in an all-pay auction and a lottery ticket that wins the good with probability $q_i(\bar{x}_i)$. Since $q_1(\bar{x}_1) = 1$, bidder 1 is effectively choosing between an all-pay auction and a "buy-it-now" option. Bidder 1 pays less than m_1 (precisely- the amount equal to the right-hand side of inequality (27)) for the "buy-it-now" option iff (27) holds as strict inequality. Also, unlike in the top auction, the all-pay auction is asymmetric. Since bidder 1 is handicapped, she gets the good in this auction when her bid exceeds bidder 2's bid by a certain margin which depends on the budgets and type distribution.[8]

[8] The mapping of bids into types is defined via the formula for the transfers $t_i(x_i) = q_i(x_i)x_i - \int_{r_i}^{x_i} q_i(s)ds$. Since $t_i(x_i)$ is strictly increasing on $[r_i, \bar{x}_i)$, a bid b_i uniquely defines x_i on this interval via $b_i = t_i(x_i)$. So, when i bids $b_i = t_i(x_i)$ and bidder j bids $b_j = t_j(x_j)$, bidder i gets the good if $\gamma_i(x_i) \geq \gamma_j(x_j)$.

Let us now consider the parties' expected payoffs in the optimal mechanism. By Lemma 10, the seller's expected revenue as a function of (m_1, m_2) is given by:

$$\pi(m_1, m_2) = \min_{\lambda \geq 0} \left\{ \int_{x \in [0,1]^2} \max\{0, \max \gamma_1(x_1), \gamma_2(x_2)\} dF(x_1) dF(x_2) + \sum_{i=1}^{2} \lambda_i m_i \right\}. \qquad (29)$$

Since $\pi(m_1, m_2)$ is a pointwise minimum in λ of a function affine in (m_1, m_2), it is concave in the vector (m_1, m_2). This fact underlies the following result:

Theorem 3 *Suppose that the aggregate budget is fixed at M, i.e., $m_1 + m_2 = M$.*[9] *Let*

$$\pi^*(M) = \max_{m_1, m_2 : m_1 + m_2 = M} \pi(m_1, m_2)$$

Then: (i) the seller gets the maximal payoff $\pi^*(M)$ when m_1 and m_2 are such that the optimal mechanism is top auction, i.e., (20) and (21) hold.

(ii) Consider two budget profiles (m_1, m_2) and (m_1', m_2') such that $m_1 + m_2 = m_1' + m_2' = M$, $m_2 > m_2'$, and the optimal mechanism under (m_1', m_2') is a budget handicap auction. Then $\pi(m_1, m_2) > \pi(m_1', m_2')$.

Theorem 3 provides an insight into the optimality of the top auction. Indeed, let us fix the aggregate budget $m_1 + m_2$ at some $M > 0$. Due to the concavity of the revenue function $\pi(.)$ in (29), on the set of such budget profiles the seller achieves the highest revenue when $m_1 = m_2 = \frac{M}{2}$. In this case, the seller offers a symmetric top auction which coincides with the optimal mechanism of Laffont and Robert (1996). If we now redistribute the aggregate budget keeping $m_1 - m_2$ small enough, the optimal mechanism is still the top auction with the same threshold and the same revenue for the seller, since these depend only on the aggregate budget in the top auction.

But when the spread $m_1 - m_2$ becomes large enough that the seller has to switch to a budget handicap auction, the seller's revenue falls and keeps decreasing as this spread increases. This is established in part (ii) of Theorem 3, and illustrated in Figure 3a.

Also, from (29) it follows that $\frac{\partial \pi(m_1, m_2)}{\partial m_i} = \lambda_i \geq 0$, so the seller's revenue strictly increases in m_i when i's budget constraint is binding. Figure 3b illustrates this under uniform type distribution, where we fix $m_2 = 0.25$ and let m_1 vary over a wide range dropping the convention that $m_1 \geq m_2$. Interestingly, the seller's revenue increase for most part comes from only one of the bidders: from bidder 1 until m_1 becomes much larger than m_2 (the seller's revenue from bidder 2 decreases in this range), and from bidder 2 when m_1 is significantly larger than m_2 so that the seller uses budget-handicap auction.

[9] To make this result non-trivial M has to be sufficiently small. In particular, we will assume that $M \leq 2\widehat{m}$.

Figure 4 illustrates, also under uniform type distribution, that a bidder's expected payoff is non-monotone in her budget. Here we drop the convention that $m_1 \geq m_2$ and consider all m_1 in $[0,1]$ with fixed $m_2 = 0.25$. Note that bidder 1's payoff decreases in m_1 when the budget-handicap auction is optimal, i.e., when m_1 is either small or large relative to m_2. This is due to two factors. First, a higher m_1 means that 1's types above the threshold pay more. The second factor is due to the handicapping effect. When $m_1 - m_2$ is large, the seller increases the degree of handicapping of bidder 1 with lower values as m_1 increases. On the other hand, when m_1 is small compared to m_2, as m_1 increases, the seller handicaps richer bidder 2 to a lesser degree, which also causes bidder 1's payoff to decrease. The bidders' payoffs are constant in m_1 when it is so large that bidder 1's budget constraint is non-binding.

In the region of optimality of the top auction, bidder 1's payoff is non-monotone in m_1 due to an interplay between two effects. The first is the direct effect of a higher payment equal to m_1 by the high types, which decreases bidder 1's expected payoff. The second effect comes from an increase in $q_1(\bar{x}_1)$, which increases bidder 1's expected payoff.

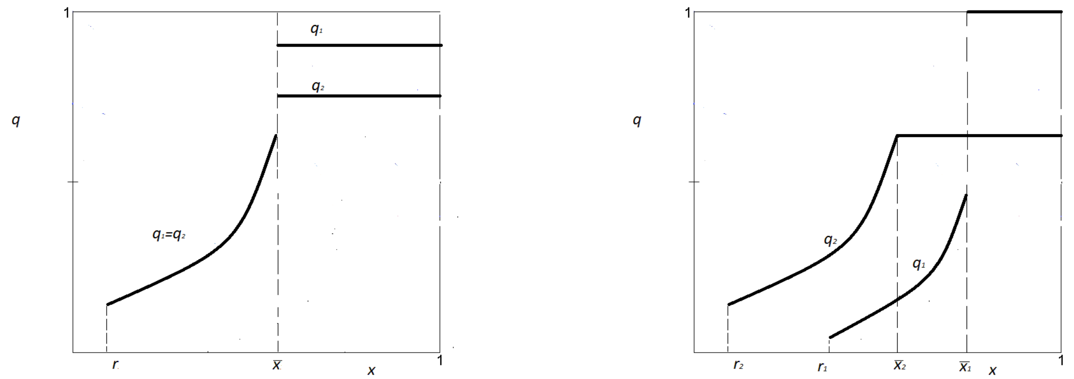
To conclude this discussion, let us provide a short analytical argument to show that a bidder's expected payoff may decrease globally as her budget increases. Specifically, consider two situations under uniform type distribution. First, suppose that $m_1 = m_2 = \epsilon > 0$, where ϵ is small. Then the optimal mechanism is a top auction, and by (20), $2\epsilon = \bar{x}^t(1+\bar{x}^t) - 2\int_{r_t}^{\bar{x}^t} x dx = \bar{x}^t + r_t^2$. So, $\epsilon < \bar{x}^t < 2\epsilon$. Since bidder 1 with value above $\bar{x}^t$ gets the good with probability $\bar{x}^t + \frac{1}{2}(1-\bar{x}^t)$ and pays ϵ, her expected payoff $v_1(\epsilon,\epsilon)$ satisfies $v_1(\epsilon,\epsilon) > (\bar{x}^t + \frac{1}{2}(1-\bar{x}^t))(1-\bar{x}^t)(E(x|x \geq \bar{x}^t) - \epsilon) > \left(\epsilon + \frac{1}{2}(1-\epsilon)\right)\left(\int_{x^t}^1 x - \epsilon dx\right) > \left(\epsilon + \frac{1}{2}(1-\epsilon)\right)\left(\frac{1}{2} - \epsilon\right)$. The last expression is close to $\frac{1}{4}$ when ϵ is small. Intuitively, the threshold x^t is low because the budget ϵ is low, and so 1 gets a fairly large surplus when her value is above x^t since she makes only a small payment equal to ϵ. So, bidder 1's expected payoff is fairly high.

Now suppose that $m_1 = \frac{1}{2}$, while $m_2 = \epsilon$. Because of competition from bidder 2, bidder 1's payoff $v_1(\frac{1}{2}, \epsilon)$ is less than the expected payoff $\frac{1}{8}$ that she gets as a single bidder with budget $\frac{1}{2}$ when the seller optimally sets the price $\frac{1}{2}$. Thus, $v_1(\epsilon,\epsilon) > v_1(\frac{1}{2},\epsilon)$ when ϵ is small.

4.1 The Case of $n > 2$ Bidders

The results of this section generalize naturally to the case of any n bidders. The details are provided in the online Appendix. Here we highlight the main features of the optimal mechanism for $n > 2$. First, the optimal mechanism is still either a top auction or a budget-handicap auction. The top auction is characterized by a common threshold for all bidders which, $\bar{x}^{t_n}$, which solves the aggregate budget constraint $\sum_{i=1,\ldots,n} m_i = \bar{x}^{t_n} \frac{1-F(\bar{x}^{t_n})^n}{1-F(\bar{x}^{t_n})} -$

Figure 1: Expected Probabilities of Trading with Two Bidders



(a) Top Auction (b) Budget-Handicap Auction

Figure 2: The Optimal Mechanism and Bidders' Budgets under Uniform Type Distribution.

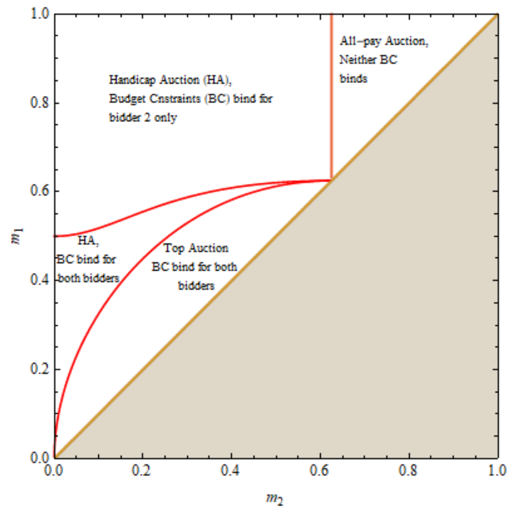
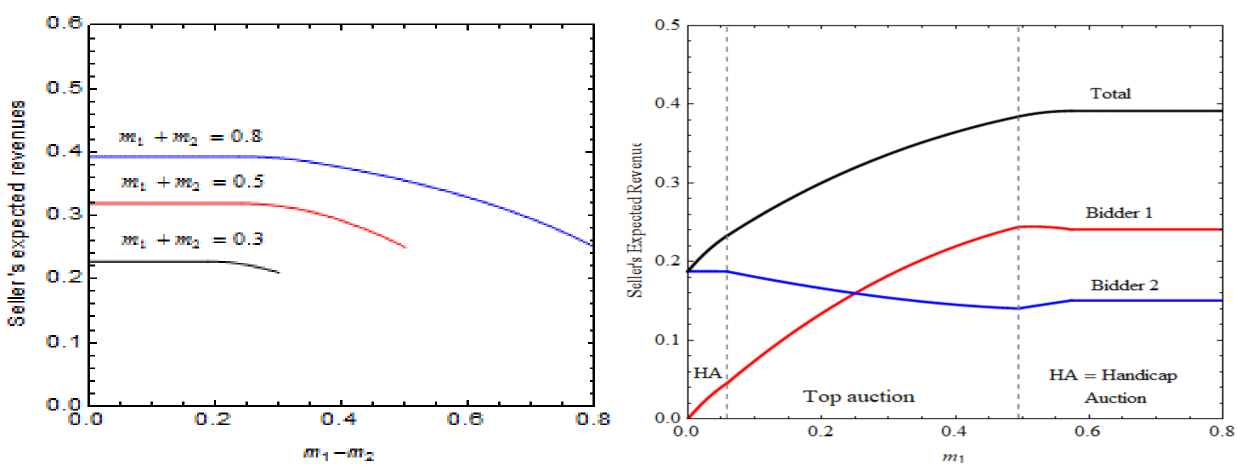
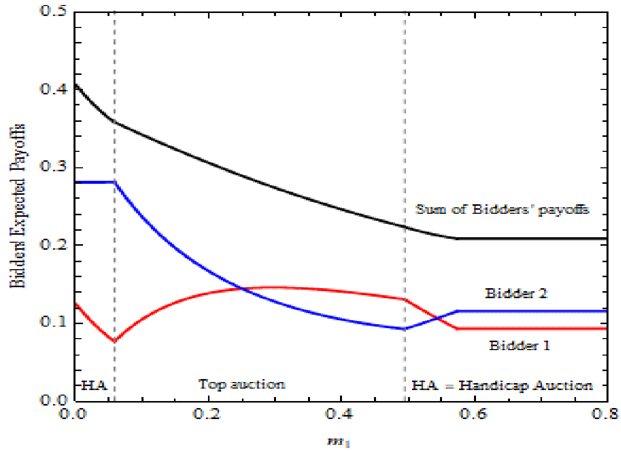


Figure 3: Seller's Expected Revenue Under Uniform Type Distribution



(a) Total Revenue (b) Revenue from Bidders 1 and 2 ($m_2 = 0.25$)

Figure 4: Bidders' Expected Payoffs Under Uniform Type Distribution ($m_2 = 0.25$)



$n \int_{r_{t_n}}^{\bar{x}^{t_n}} F(s)^{n-1} ds$, where the "reservation value" r_{t_n} is related to $\bar{x}^{t_n}$ in the same way as in the case of 2 bidders. Discrimination between the bidders is limited to the types in $[\bar{x}^{t_n}, 1]$: the probability that bidder i with type in $[\bar{x}^{t_n}, 1]$ gets the good is increasing in her budget.

Similarly to the case of 2 bidders, the top auction is an optimal mechanism if and only if $\bar{x}^{t_n}$ satisfies a condition under which it is feasible to have all budget constraints bind "at the top." But since we now have more bidders, this condition involves a system of $n-1$ inequalities requiring that the average budget of the richest k bidders do not exceed the average budget of the poorest $n-k$ bidders by too much, for all $k = 1, ..., n-1$.

If any of these inequalities fail, the optimal mechanism is an n-bidder budget handicap auction with different thresholds for different bidders or groups of bidders. There may exist clusters of bidders with the same threshold, but not all bidders are in the same cluster. As in the two bidder case, a richer bidder has a (weakly) higher threshold and a strictly higher probability of trading when her value exceeds this threshold than a poorer bidder. At lower valuations, the seller discriminates in favor of poorer bidders with lower thresholds, but does not discriminate between the bidders in the same cluster (who have the same threshold).

The challenging part in computing "budget-handicap" auction for n bidders is determining "clusters" of bidders with the same threshold. This problem is not analytically difficult as it only involves checking the first-order conditions for different cluster configurations. However, one may have to go through all such configurations to find the optimal one. In the online Appendix we show that with three bidders under uniform type distribution every possible cluster configuration is optimal for a set of budgets of a positive measure.

5 The Constrained-Efficient Mechanism

In this section, we characterize the constrained efficient mechanism which maximizes the expected social surplus, $\sum_{i=1}^{n}\int_0^1 q_i(x_i)x_i dF(x_i)$, subject to the incentive and individual rationality constraints, budget constraints and feasibility constraints (2)-(5), respectively.

In the standard setting, a VCG mechanism attains full efficiency under private values. But with budget constraints, VCG mechanism does not work since the bidders' willingness to pay cannot be fully translated into their bids. High-value bidder types can no longer afford to pay the value of the externality that they impose on the others.[10]

The constrained-efficient mechanism is qualitatively similar to the optimal mechanism, so we will omit the details of the proofs and present the qualitative characterization and the differences between these two mechanisms. Repeating the steps of the analysis of the optimal mechanism and letting $\bar{x}_i^e$ denote the threshold value at which bidder i's transfer reaches its maximum, and λ_i^e denote the multiplier associated with i's budget constraint yields the following Lagrangian:

$$\mathcal{L}^e = \sum_{i=1}^{n}\int_0^1 q_i(x_i)x_i dF(x_i) - \lambda_i^e\left(q_i(\overline{x}_i^e)\overline{x}_i^e - \int_0^{\overline{x}_i^e} q_i(x)dx - m_i\right) =$$

$$\sum_{i=1}^{n}\left(\int_0^{\bar{x}_i^e} q_i(x_i)\left(x_i + \frac{\lambda_i^e}{f(x_i)}\right)dF(x_i) + \int_{\bar{x}_i^e}^{1} q_i(\bar{x}_i)\left(E(x_i|x_i \geq \bar{x}_i^e) - \frac{\lambda_i^e\overline{x}_i^e}{1-F(\bar{x}_i^e)}\right)dF(x_i) + \lambda_i^e m_i\right) \tag{30}$$

Applying Lemma 4 to (30) we obtain the same modified objective (11) for the constrained-efficiency problem as for the optimal mechanism, albeit with the new virtual values:

$$\gamma_i^e(x_i) = \begin{cases} x_i + \frac{\lambda_i^e}{f(x_i)}, & \text{if} \quad x_i < \overline{x}_i^e, \\ E(x_i|x_i \geq \bar{x}_i^e) - \frac{\lambda_i^e\overline{x}_i^e}{1-F(\overline{x}_i^e)}, & \text{if} \quad x_i \geq \overline{x}_i^e. \end{cases} \tag{31}$$

The virtual values $\gamma_i^e(.)$ are different from $\gamma_i(.)$ in the optimal mechanism, since in the constrained-efficient mechanism the designer maximizes total surplus and not just his profits.

To ensure that the counterparts of the results in Section 3 hold here we need the following Assumption, which replaces Assumption 1 (monotone hazard rate) in this section:

Assumption 2 *The density $f(.)$ is log-concave, and satisfies $\frac{f'(0)}{f(0)} \leq \frac{1}{\mathbf{E}x}$.*

[10] One way to achieve efficiency is to subsidize the bidders. Yet, in many real-world situations subsidization is infeasible or politically unacceptable, and so we look for a mechanism that does not rely on it.

Following the same arguments as in Lemmas 5 and 8, we can show that under Assumption 2 the vectors of thresholds $(\bar{x}_1^e, ..., \bar{x}_n^e)$ and multipliers $(\lambda_1^e, ..., \lambda_n^e)$ satisfy the following relationship in the constrained efficient mechanism. For $i \in \{2, ..., n\}$ and for $i = 1$ if $\bar{x}_1^e = \bar{x}_2^e$:

$$\lambda_i^e = \frac{\int_{\bar{x}_i^e}^1 1 - F(x)dx}{\bar{x}_i^e + (1 - F(\bar{x}_i^e)/f(\bar{x}_i^e)}. \tag{32}$$

When the highest threshold $\bar{x}_1$ is such that $\bar{x}_1 > \bar{x}_2$, then:

$$\lambda_1^e = f(\bar{x}_1^e)(\gamma_2^e(\bar{x}_2^e) - \bar{x}_1^e). \tag{33}$$

It is easy to see that, under Assumption 2, λ_i^e is decreasing in $\bar{x}_i^e$ and the virtual value $\gamma_i^e(.)$ is increasing on $[0, \bar{x}_i^e]$ for all i. The latter implies that Lemmas 6, 7 and 10 apply to the constrained-efficient mechanism verbatim. So, we can derive this mechanism by minimizing the dual function $g^e(.)$ defined similarly to the dual function $g(.)$ in (19), with the only difference that $g^e(.)$ involves the virtual values $\gamma_i^e(.)$ in (31) and thresholds $\bar{x}^e(\gamma^e)$ given by (32) and (33), rather than $\gamma_i(.)$ in (12) and $\bar{x}(\gamma)$ given by (16) and (17), respectively.

The details of the analysis are similar to the ones for the optimal mechanism and are omitted. Focussing again on the case of two bidders, the budget constraint of the poorer bidder 2 is binding at threshold $\bar{x}_2^e < 1$ if $m_2 < \breve{m} \equiv 1 - \int_0^1 F(s)ds$[11], and the constrained-efficient mechanism is also either a top auction or a budget-handicap auction. These mechanisms are presented in the next two Theorems, which are counterparts of Theorems 1 and 2.

Theorem 4 *Suppose that $m_2 < \breve{m}$. The constrained-efficient mechanism is a top auction in which the two bidders have a common threshold $\bar{x}^{te}$ if and only if the following conditions hold:*

$$m_1 + m_2 = \bar{x}^{te}\left(1 + F(\bar{x}^{te})\right) - 2\int_0^{\bar{x}^{te}} F(s)ds, \tag{34}$$

$$m_1 - m_2 \leq \bar{x}^{te}(1 - F(\bar{x}^{te})). \tag{35}$$

The expected trading probabilities are $q_i(x_i) = F(x_i)$ for all $x_i \in [0, \bar{x}^{te})$, and $q_i(\bar{x}^{te})$ uniquely defined by the budget constraints:

$$m_i = \bar{x}^{te} q_i(\bar{x}^{te}) - \int_0^{\bar{x}^{te}} F(s)ds, \quad i \in \{1, 2\}.$$

[11] Recall that the budget constraint of bidder 2 is binding in the optimal mechanism at some $\bar{x}_2 < 1$ if $m_2 < \widehat{m} = 1 - \int_{r:r-\frac{1-F(r)}{f(r)}=0}^1 F(s)ds$ and observe that $\breve{m} < \widehat{m}$.

The constrained-efficient top auction is not fully efficient because both bidders' types in $[\bar{x}^{te}, 1]$ are tied, and so the good is allocated inefficiently between them. Equation (34) implies that $\bar{x}^{te}$ is increasing in $m_1 + m_2$. Also, the reservation value in the constrained efficient top auction is zero, while in the optimal top auction it is positive. For this reason, under the same budget profile the threshold in the optimal auction $\bar{x}^t$ is lower, i.e., $\bar{x}^t < \bar{x}^{te}$. Notably, the budget sets (m_1, m_2) under which the top auction is constrained-efficient mechanism and under which it is optimal mechanism are non-nested. This is so because the right-hand sides of (21) and (35) which specify the upper bounds on $m_1 - m_2$ in the two cases are non-monotone in the respective thresholds. We illustrate this non-nestedness in the online Appendix (Section 3B) for the case of uniform type distributions.

When conditions (34) and (35) fail to hold, then the constrained-efficient mechanism is a budget handicap auction. Specifically, we have the following counterpart of Theorem 2:

Theorem 5 *Suppose that $m_2 < \check{m}$ and conditions (34)-(35) do not hold. Then the constrained efficient mechanism is a budget-handicap auction in which $\bar{x}_1^e > \bar{x}_2^e$.*

The probabilities of trading $q_i(x)$, $i \in \{1, 2\}$ in this mechanism satisfy: $q_1(\bar{x}_1^e) = 1$, $q_2(\bar{x}_2^e) = F(\bar{x}_1^e)$; $q_1(x) = 0$ for all $x \in [0, r_1)$, $q_1(x) < F(x)$ for all $x \in [r_1, \bar{x}_1^e)$ and $q_2(x) > F(x)$ for all $x \in [0, \bar{x}_2^e)$ where $r_1 > 0$.[12]

The optimal thresholds $\bar{x}_1^e$ and $\bar{x}_2^e$ are determined as follows:

(i) $\bar{x}_1^e$ and $\bar{x}_2^e$ are the unique solution to the following system of equations:

$$m_2 = \bar{x}_2^e F(\bar{x}_1^e) - \int_0^{\bar{x}_2^e} q_2(x_2)dx_2, \tag{36}$$

$$\bar{x}_1^e = \bar{x}_2^e + \frac{\int_{\bar{x}_2^e}^1 1 - F(x)dx}{f(\bar{x}_2^e)\bar{x}_2^e + (1 - F(\bar{x}_2^e)}, \tag{37}$$

provided that this solution is such that bidder 1's budget constraint holds, i.e.,

$$m_1 \geq \bar{x}_1^e - \int_0^{\bar{x}_1^e} q_1(x_1)dx_1. \tag{38}$$

(ii) Otherwise, i.e., if the solution to (36) and (37) does not satisfy (38), then $\bar{x}_1^e$ and $\bar{x}_2^e$ are the unique solution to the system of binding budget constraints that consists of (36) and:

$$m_1 = \bar{x}_1^e - \int_0^{\bar{x}_1^e} q_1(x_1)dx_1. \tag{39}$$

[12] The reservation value r_1 is uniquely defined by the following equation: $r_1 + \frac{f(\bar{x}_1^e)}{f(r_1)}\left(\bar{x}_2^e + \frac{\int_{\bar{x}_2^e}^1 1-F(x)dx}{\bar{x}_2^e f(\bar{x}_2^e)+(1-F(\bar{x}_2^e)} - \bar{x}_1^e\right) = \frac{\int_{\bar{x}_2^e}^1 1-F(x)dx}{f(0)(\bar{x}_2^e+(1-F(\bar{x}_2^e)/f(\bar{x}_2^e))}$.

As in the case of the optimal auction, part (i) of Theorem 5 characterizes the case in which only the budget constraint of bidder 2 is binding, while part (ii) involves the case where both bidders' budget constraints are binding. The reservation value of bidder 1 (2) is positive (zero) in the budget-handicap auction because her virtual value is positive for all x_1 (x_2) but $\lambda_2 > \lambda_1$, as follows from (31), (32) and (33).

The budget-handicap auction is less efficient than the top auction due to a greater degree misallocation in it. Particularly, Theorem 3 applies here verbatim implying that under a fixed aggregate budget, $m_1 + m_2$, the maximal efficiency is attained when the difference $m_1 - m_2$ is small enough, so that the constrained-efficient mechanism is a top auction. The efficiency level in the top auction depends only on the aggregate budget. However, when $m_1 - m_2$ is sufficiently large that (35) fails, the constrained efficient mechanism takes a form of a budget handicap auction and the efficiency level decreases. The efficiency in the budget handicap auction keeps falling as $m_1 - m_2$ increases.

6 Conclusions

In this paper we have characterized the optimal and constrained-efficient mechanisms for bidders with commonly known and unequal budgets. Qualitatively these mechanisms belong either to the class of "top auctions" or "budget-handicap" auctions. A top-auction is optimal and constrained-efficient when budget differences are small. It discriminates between the bidders only when they have high valuations and favors high-value bidders. When budget differences are sufficiently large, the seller uses a "budget-handicap" auction in which she also discriminates between bidders with low valuations favoring low-budget bidders. This feature of the budget-handicap auction provides justification for favoring smaller or minority-owned businesses in public procurement and other mechanisms.

Our mechanisms have features of an all-pay auction, since a bidder always pays her bid. It would be interesting to study mechanisms in which a bidder pays only when (s)he gets the good. We leave this issue for future research.

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7 Appendix A: Proofs

Proof of Lemma 3: Since $q_i(.)$ is increasing on $[0,1]$ by Lemma 2, $t_i(.)$ must also be increasing by incentive compatibility. Now, suppose that $t_i(x_i') < t_i(1)$ for some $x_i' \in (\bar{x}_i, 1)$. Then by definition of $\bar{x}_i$, there exists $x_i'' \in [\bar{x}_i, x_i')$ s.t. $t_i(x_i'') > t_i(x_i')$, which contradicts that $t_i(.)$ is increasing. So, we must have $t_i(x_i) = t_i(1)$ and hence $q_i(x_i) = q_i(1)$ for all $x_i \in (\bar{x}_i, 1]$.

Since $(q_i(1), t_i(1))$ satisfies incentive and individual rationality constraints (2) and (3) of any type $x_i > \bar{x}_i$, by continuity it is also incentive compatible and individually rational for type $\bar{x}_i$. So we can WLOG take $t_i(\bar{x}_i) = t_i(1)$ and $q_i(\bar{x}_i) = \bar{q}_i$, because changing the allocation of a single agent type does not affect the seller's expected profits from the mechanism. *Q.E.D.*

Proof of Lemma 5: The proof of Lemma 5 relies on the following Lemma:

Lemma 11 *The following cannot hold in any solution to problem (14) for any i s.t. $\bar{x}_i < 1$:*

(a) $\gamma_i^-(\bar{x}_i) > \gamma_i(\bar{x}_i)$, and the set $A_i(\lambda, \bar{\mathbf{x}})= \{x_{-i} \in [0,1]^{n-1} : \gamma_i(\bar{x}_i) \leq \max\{0, \max_{j\neq i} \gamma_j(x_j)\} < \gamma_i^-(\bar{x}_i)\}$ has a positive measure, where $(\lambda, \bar{\mathbf{x}})= (\lambda_1, ..., \lambda_n, \bar{x}_1, ..., \bar{x}_n)$.

(b) $\gamma_i(\bar{x}_i) > \gamma_i^-(\bar{x}_i)$, and the set $B_i(\lambda, \bar{\mathbf{x}})= \{x_{-i} \in [0,1]^{n-1} : \gamma_i^-(\bar{x}_i) < \max\{0, \max_{j\neq i} \gamma_j(x_j)\} \leq \gamma_i(\bar{x}_i)\}$ has a positive measure.

Proof of Lemma 11: We start by establishing several useful properties of $\mathcal{L}(\bar{x}, \lambda)$. First, $\mathcal{L}(\bar{x}, \lambda)$ is convex in λ. Indeed, since $\max\{0, \max_i\{\gamma_i(x_i)\}\}$ is convex in $(\gamma_1(x_1), ..., \gamma_n(x_n))$ and $\gamma_i(x_i)$ is linear in λ_i, $\max\{0, \max_i\{\gamma_i(x_i)\}\}$ is convex in λ . Integration preserves convexity of the integrand $\max\{0, \max_i\{\gamma_i(x_i)\}\} \prod_i f(x_i)$ in λ, so $\mathcal{L}(\bar{x}, \lambda)$ is convex in λ.

Second, $\mathcal{L}(\bar{x}, \lambda)$ is continuous in $(\bar{x}, \lambda)$, and is strictly increasing in λ_i when $\lambda_i > 1 - F(\bar{x}_i)$. To establish the last claim, suppose that $\lambda_i > 1 - F(\bar{x}_i)$ for some i, and so $\gamma_i(\bar{x}_i) < 0 < \gamma_i^-(\bar{x}_i)$. Then the derivative of $\mathcal{L}(\bar{x}, \lambda)$ with respect to λ_i exists and is equal to:

$$\frac{\partial \mathcal{L}(\bar{x}, \lambda)}{\partial \lambda_i} = m_i + \int_{x_i \in [0,\bar{x}_i), x_{-i} \in [0,1]^{n-1} : \gamma_i(x_i) > \max\{0, \max_{j\neq i} \gamma_j(x_j)\}\}} \frac{\partial \gamma_i(x_i)}{\partial \lambda_i} dF(x_i) d \prod_{j\neq i} F(x_j) > 0 \quad (40)$$

(40) is strictly positive because $\frac{\partial \gamma_i(x_i)}{\partial \lambda_i} = \frac{1}{f(x_i)} > 0$ when $x_i \in [0, \bar{x}_i)$. So, for any $\bar{x} \in [0,1]^n$, $\arg\min_{\lambda \geq 0} \mathcal{L}(\bar{x}, \lambda)$ exists and satisfies $\lambda_i \leq 1 - F(\bar{x}_i)$ for all i. Therefore, the primal function $p(\bar{x}) = \min_{\lambda \geq 0} \mathcal{L}(\bar{x}, \lambda)$ is well-defined for all $\bar{x} \in [0,1]^n$ and is continuous by Berge's Maximum Theorem, so $\arg\max_{\bar{x} \in [0,1]} p(\bar{x})$ exists. By Danskin Theorem (see e.g. Bertsekas (2001), p.131) the right-hand side derivative $\frac{\partial_+ p(\bar{x})}{\partial \bar{x}_i}$ exists and satisfies $\frac{\partial_+ p(\bar{x}_i, \bar{x}_{-i})}{\partial \bar{x}_i} = \frac{\partial_+ \mathcal{L}(\bar{x}, \lambda^m)}{\partial \bar{x}_i}$ where $\lambda^m = \arg\min_{\lambda \geq 0} \mathcal{L}(\bar{x}, \lambda)$.[13] Therefore, $(\bar{x}, \lambda)$ cannot be a solution to problem (14) if $\bar{x}_i < 1$ and $\frac{\partial_+ \mathcal{L}(\bar{x}, \lambda)}{\partial \bar{x}_i} > 0$. So, to complete the proof of the Lemma we will show that $\frac{\partial_+ \mathcal{L}(\bar{x}, \lambda)}{\partial \bar{x}_i} > 0$ under conditions (a) and (b).

To this end, let us differentiate (13) to obtain:

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}} \left(\max\{0, \gamma_i^-(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\} - \max\{0, \gamma_i(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\} \right) dF(x_{-i})$$
$$+ \int_{x \in [0,1]^n} \frac{\partial_+ \max\{0, \max_{j=1,...,n} \gamma_j(x_j)\}}{\partial \bar{x}_i} dF(x). \quad (41)$$

The first term in (41) comes from possible discontinuity of the integrand of $\mathcal{L}(\bar{x}, \lambda)$ in (13) at $x_i = \bar{x}_i$. The second term comes from differentiating the integrand of $\mathcal{L}(\bar{x}, \lambda)$.

Suppose that the conditions in part (a) of the Lemma hold, i.e., $\gamma_i^-(\bar{x}_i) > \gamma_i(\bar{x}_i)$ and the

[13] Although $\mathcal{L}(\bar{x}, \lambda)$ may not possess a derivative with respect to $\bar{x}_i$ because it contains max operator, it does possess left- and right-hand derivatives, $\frac{\partial_- \mathcal{L}(\bar{x}, \lambda)}{\partial \bar{x}_i}$ and $\frac{\partial_+ \mathcal{L}(\bar{x}, \lambda)}{\partial \bar{x}_i}$, respectively.

set $A_i(\lambda, \bar{\mathbf{x}})$ has a positive measure. Then the first term in (41) can be rewritten as follows:

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \max\{0, \max_{j \neq i} \gamma_j(x_j)\} < \gamma_i(\bar{x}_i)} \gamma_i^-(\bar{x}_i) - \gamma_i(\bar{x}_i) dF(x_{-i}) +$$
$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \gamma_i(\bar{x}_i) \leq \max\{0, \max_{j \neq i} \gamma_j(x_j)\} < \gamma_i^-(\bar{x}_i)} \left(\gamma_i^-(\bar{x}_i) - \max\{0, \max_{j \neq i} \gamma_j(x_j)\} \right) dF(x_{-i}) \tag{42}$$

Now, let us consider the second term of (41). From (12) we have:

$$\frac{\partial_+ \gamma_i(x_i)}{\partial \bar{x}_i} = \begin{cases} 0, & \text{if } x_i < \bar{x}_i, \\ 1 - \frac{\lambda_i}{(1-F(\bar{x}_i))^2} \left(1 - F(\bar{x}_i) + \bar{x}_i f(\bar{x}_i)\right) = \frac{f(\bar{x}_i)}{1-F(\bar{x}_i)} \left(\gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i)\right), & \text{if } x_i \geq \bar{x}_i, \end{cases} \tag{43}$$

From (43) and $\gamma_i^-(\bar{x}_i) > \gamma_i(\bar{x}_i)$ it follows that $\frac{\partial_+ \gamma_i(x_i)}{\partial \bar{x}_i} < 0$ for $x_i \geq \bar{x}_i$. This and the fact that $\frac{\partial \gamma_j(x_j)}{\partial \bar{x}_i} = 0$ for $j \neq i$ imply that the second term in (41) equals: $f(\bar{x}_i) \int_{x_{-i}: \max\{0, \max_{j \neq i} \gamma_j(x_j)\} < \gamma_i(\bar{x}_i)} \gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i) dF(x_{-i})$. Using this and (42) in (41) yields:

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = f(\bar{x}_i) \int_{x_{-i}: \gamma_i(\bar{x}_i) \leq \max\{0, \max_{j \neq i} \gamma_j(x_j)\} < \gamma_i^-(\bar{x}_i)} \left(\gamma_i^-(\bar{x}_i) - \max\{0, \max_{j \neq i} \gamma_j(x_j)\} \right) dF(x_{-i}) > 0 \tag{44}$$

where the inequality holds because the set of integration is $A_i(\lambda, \bar{\mathbf{x}})$, which has a positive measure, and the integrand is positive everywhere on this set.

Next, consider part (b). Again, the proof is by contradiction, so suppose that $\gamma_i(\bar{x}_i) > \gamma_i^-(\bar{x}_i)$, and the set $B_i(\lambda, \bar{\mathbf{x}})$ has a positive measure. The former is equivalent to $\lambda_i < \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$ and implies that $\gamma_i(\bar{x}_i) > 0$. Then the first term in (41) is equal to:

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \max_{j \neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \left(\max\{0, \gamma_i^-(\bar{x}_i), \max_{j \neq i} \gamma_j(x_j)\} - \gamma_i(\bar{x}_i) \right) dF(x_{-i}) \tag{45}$$

From (43) it follows that $\frac{\partial_+ \gamma_i(x_i)}{\partial \bar{x}_i} > 0$ if $x_i \geq \bar{x}_i$ and $\frac{\partial_+ \gamma_i(\bar{x}_i)}{\partial \bar{x}_i} = 0$ if $x_i < \bar{x}_i$. Since $\frac{\partial \gamma_j(x_j)}{\partial \bar{x}_i} = 0$ for $j \neq i$ and $\gamma_i(\bar{x}_i) > 0$, the second term of (41) in this case equals: $f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \max_{j \neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i) dF(x_{-i})$. Combining this with (45) yields:

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \max_{j \neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \max\{0, \gamma_i^-(\bar{x}_i), \max_{j \neq i} \gamma_j(x_j)\} - \gamma_i^-(\bar{x}_i) dF(x_{-i}) =$$
$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \gamma_i^-(\bar{x}_i) < \max\{0, \max_{j \neq i} \gamma_j(x_j)\} \leq \gamma_i(\bar{x}_i)} \max\{0, \max_{j \neq i} \gamma_j(x_j)\} - \gamma_i^-(\bar{x}_i) dF(x_{-i}) > 0. \tag{46}$$

where the equality holds because the first integrand is nonnegative for all x_{-i} and is positive only if $\gamma_i^-(\bar{x}_i) < \max\{0, \max_{j\neq i}\gamma_j(x_j)\}$, and also $\{x_{-i} : \max\{0, \max_{j\neq i}\gamma_j(x_j)\} \leq \gamma_i(\bar{x}_i)\} = \{x_{-i} : \max_{j\neq i}\gamma_j(x_j) \leq \gamma_i(\bar{x}_i)\}$ because $\gamma_i(\bar{x}_i) > 0$. Finally, the set of integration under the last integral is $B_i(\lambda, \bar{\mathbf{x}})$ which has a positive measure, and the integrand is positive for any x_{-i} in this set, so (46) is strictly positive, and so we cannot be at the solution to our problem. This completes the proof of the Lemma 11. *Q.E.D.*

Next, we prove several claims regarding the solution to problem (14).

Claim 1: $\lambda_i \leq 1 - F(\bar{x}_i)$ for all $i \in \{1, ..., n\}$. This claim holds because $\mathcal{L}(\bar{x}, \lambda_i)$ is strictly increasing in λ_i when $\lambda_i > 1 - F(\bar{x}_i)$, as shown in expression (40) above.

Claim 1 has four important implications: (i) for all i, $\gamma_i(x_i)$ is increasing on $[0, \bar{x}_i)$ (ii) We can restrict the domain of $(\lambda_1, ..., \lambda_n)$ to $[0,1]^n$; (iii) $\gamma_i(\bar{x}_i) \geq 0$; (iv) for all i, there exists $\delta \in (0,1]$ s.t. $\gamma_i(x_i) \leq 0$ for all $x_i \in [0, \delta]$.

Claim 2: Suppose that $\gamma_i^-(\bar{x}_i) \leq \gamma_j^-(\bar{x}_j)$ for some i, j and $\bar{x}_i < 1$. Then $\gamma_i^-(\bar{x}_i) \leq \gamma_i(\bar{x}_i)$.

Suppose that $\gamma_i(\bar{x}_i) < \gamma_i^-(\bar{x}_i)$. Then by Claim 1 $\gamma_i^-(\bar{x}_i) > 0$, and also $\gamma_k(0) \leq 0$ for all k. So there exists $\tilde{x}_k \in (0,1]$ s.t. $\gamma_k(x_k) < \gamma_i^-(\bar{x}_i)$ for all $x_k \in [0, \tilde{x}_k)$, and there exist x_j^1, x_j^2, $x_j^1 < x_j^2$, s.t. $\gamma_i(\bar{x}_i) < \gamma_j(x_j) < \gamma_i^-(\bar{x}_i)$ for all $x_j \in [x_j^1, x_j^2]$. So, $\gamma_i(\bar{x}_i) < \max\{0, \gamma_j(x_j), \max_{k\neq i,j}\gamma_k(x_k)\} < \gamma_i^-(\bar{x}_i)$ if $x_k \in [0, \tilde{x}_k)$, $k \notin \{i, j\}$, and $x_j \in [x_j^1, x_j^2]$. So, the set $A_i(\lambda, \bar{\mathbf{x}})$ has a positive measure, and by Lemma 11 we are not at the optimum.

Claim 3: Suppose that $\gamma_i^-(\bar{x}_i) < \gamma_j^-(\bar{x}_j)$ for some i, j and $\bar{x}_i < 1$. Then $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$.

By Claim 2 $\gamma_i^-(\bar{x}_i) \leq \gamma_i(\bar{x}_i)$. Next, suppose that $\gamma_i^-(\bar{x}_i) < \gamma_i(\bar{x}_i)$. Then $\lambda_i < 1 - F(\bar{x}_i)$ and so $\gamma_i(\bar{x}_i) \geq 0$. Then by an argument similar to that in Claim 2 the set $B_i(\lambda, \bar{\mathbf{x}})$ has a positive measure. So, $\gamma_i^-(\bar{x}_i) < \gamma_i(\bar{x}_i)$ cannot be optimal by Lemma 11.

Claim 4. Suppose that $\gamma_i^-(\bar{x}_i) = \gamma_j^-(\bar{x}_j)$ and $\gamma_i(\bar{x}_i) \leq \gamma_j(\bar{x}_j)$. Then $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$.

If $\bar{x}_i = 1$, then the claim holds since $\lambda_i = 0$ by Claim 1. Now suppose that $\bar{x}_i < 1$. Then $\gamma_i^-(\bar{x}_i) \leq \gamma_i(\bar{x}_i)$ by Claim 2. To rule out $\gamma_i^-(\bar{x}_i) < \gamma_i(\bar{x}_i)$, suppose that this is so. By Claim 1, $\gamma_j(\bar{x}_j) \geq 0$. Also, $\gamma_i^-(\bar{x}_i) = \gamma_j^-(\bar{x}_j)$, $\bar{x}_i < 1$ and Claim 1 imply that $\bar{x}_j < 1$. So, for $k \notin \{i, j\}$ there is $\delta_k > 0$ s.t. $[0, \delta_k] \subseteq \{x_k : \gamma_k(x_k) \leq \gamma_j(\bar{x}_j)\}$, and $\{x_i : \gamma_j^-(\bar{x}_j) < \gamma_i(x_i) \leq \gamma_j(\bar{x}_j)\} = [\bar{x}_i, 1]$. So, set $B_j(\lambda, \bar{\mathbf{x}})$ has a positive measure, so we are not at the optimum by Lemma 11.

Claim 5. Suppose that $\gamma_i^-(\bar{x}_i) > \gamma_j^-(\bar{x}_j)$ for some i and j. Then $\gamma_i(\bar{x}_i) > \gamma_j^-(\bar{x}_j)$.

Note that $\gamma_i^-(\bar{x}_i) > \gamma_j^-(\bar{x}_j)$ implies that $\bar{x}_j < 1$. So, by Claim 3, $\gamma_j^-(\bar{x}_j) = \gamma_j(\bar{x}_j)$. Now suppose that $\gamma_i(\bar{x}_i) \leq \gamma_j^-(\bar{x}_i) = \gamma_j(\bar{x}_j)$. Then $\bar{x}_i < 1$ and $\gamma_i^-(\bar{x}_i) > 0$, since $\gamma_i(\bar{x}_i) \geq 0$ by Claim 1. Claim 1 also implies that for all $k \notin \{i, j\}$ there exists $\tilde{x}_k \in (0,1]$ s.t. $\gamma_k(x_k) < \gamma_i^-(\bar{x}_i)$ for all $x_k \in [0, \tilde{x}_k)$. So both the set $\{x_j : \gamma_i(\bar{x}_i) \leq \gamma_j(x_j) < \gamma_i^-(\bar{x}_i)\}$ and the set $A_i(\lambda, \bar{\mathbf{x}})$ have a positive measure, and by Lemma 11 we cannot be at the optimum.

Claim 6. If $\bar{x}_j = \bar{x}_i$ for some i, j, then $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i) = \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$.

If $\bar{x}_j = \bar{x}_i = 1$, then the claim follows from Claim 1. So, suppose that $\bar{x}_j = \bar{x}_i < 1$. First, suppose that $\gamma_j^-(\bar{x}_j) > \gamma_i^-(\bar{x}_i)$. Then from definition (12) it follows that $\lambda_j > \lambda_i$, and hence $\gamma_i(\bar{x}_i) > \gamma_j(\bar{x}_j)$ by Claim 1. By Claim 3 $\gamma_i(\bar{x}_i) = \gamma_i^-(\bar{x}_i)$. So, $\gamma_j^-(\bar{x}_j) > \gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i) > \gamma_j(\bar{x}_j)$, which contradicts Claim 5. Thus, we must have $\gamma_j^-(\bar{x}_j) = \gamma_i^-(\bar{x}_i)$. Then by Claim 4, WLOG, $\gamma_j(\bar{x}_j) \geq \gamma_j^-(\bar{x}_j) = \gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$.

Finally, $\gamma_j^-(\bar{x}_j) = \gamma_i^-(\bar{x}_i)$ and $\bar{x}_j = \bar{x}_i$ imply that $\lambda_j = \lambda_i$. So, $\gamma_j(\bar{x}_j) = \gamma_i(\bar{x}_i)$.

Claim 7. If $\bar{x}_j > \bar{x}_i$, then either $\min\{\gamma_j^-(\bar{x}_j), \gamma_j(\bar{x}_j)\} > \gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$, or $\gamma_j(\bar{x}_j) > \gamma_j^-(\bar{x}_j) = \gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$.

Let us show that $\gamma_j^-(\bar{x}_j) \geq \gamma_i^-(\bar{x}_i)$. For suppose otherwise. Then $\bar{x}_j < 1$ and by Claims 3 and 5, $\min\{\gamma_i^-(\bar{x}_i), \gamma_i(\bar{x}_i)\} > \gamma_j^-(\bar{x}_j) = \gamma_j(\bar{x}_j)$. However, by (12) this contradicts $\bar{x}_j > \bar{x}_i$.

If $\gamma_j^-(\bar{x}_j) > \gamma_i^-(\bar{x}_i)$, then $\min\{\gamma_j^-(\bar{x}_j), \gamma_j(\bar{x}_j)\} > \gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$ by Claims 3 and 5.

If $\gamma_i^-(\bar{x}_i) = \gamma_j^-(\bar{x}_j)$, then (12) and $\bar{x}_j > \bar{x}_i$ imply that $\lambda_j < \lambda_i$, and hence $\gamma_i(\bar{x}_i) < \gamma_j(\bar{x}_j)$. So, by Claim 4, $\gamma_i^-(\bar{x}_j) = \gamma_i(\bar{x}_i) = \gamma_j^-(\bar{x}_j) < \gamma_j(\bar{x}_j)$. This completes the proof of Claim 7.

Part 1 of the Theorem then follows from Claims 6 and 7 once we observe that, by (12), $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$ is equivalent to $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i) = \frac{\bar{x}_i^2 f(\bar{x}_i)}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$ and $\lambda_i = \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$.

To establish Part 2 of the Theorem, suppose that $\bar{x}_{h_1} > \bar{x}_i$ for all $i \neq h_1$. So by Claim 7, either $\min\{\gamma_{h_1}^-(\bar{x}_{h_1}), \gamma_{h_1}(\bar{x}_{h_1})\} > \gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$, or $\gamma_{h_1}(\bar{x}_{h_1}) > \gamma_{h_1}^-(\bar{x}_{h_1}) = \gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$.

However, $\min\{\gamma_{h_1}(\bar{x}_{h_1}), \gamma_{h_1}^-(\bar{x}_{h_1})\} > \gamma_i(\bar{x}_i) = \gamma_i^-(\bar{x}_i)$ for all $i \neq h_1$ contradicts the definition of $\bar{x}_{h_1}$ in (7). Indeed, let $\bar{x}'_{h_1}$ be such that $\bar{x}'_{h_1} - \frac{1-\lambda_{h_1}-F(\bar{x}'_{h_1})}{f(\bar{x}'_{h_1})} = \max_{i \neq h_1} \gamma_i(\bar{x}_i)$. Note that $\bar{x}'_{h_1} < \bar{x}_{h_1}$ because $\gamma_{h_1}(0) \leq 0 < \max_{i \neq h_1} \gamma_i(\bar{x}_i) < \gamma_{h_1}^-(\bar{x}_{h_1})$. Then for all $x > \bar{x}'_{h_1}$, we have $\gamma_{h_1}(x) > \max_{j \neq i} \max_{x_i \in [0,1]} \gamma_i(x_i)$ and $q_{h_1}(x) = 1$ and so $t_{h_1}(x) = t_{h_1}(1)$, contradicting (7).

So, let us change the threshold of bidder h_1 from $\bar{x}_{h_1}$ to $\bar{x}'_{h_1}$. The value of $\mathcal{L}(.)$ in (13) remains the same after this, i.e., $\mathcal{L}(\bar{x}, \lambda) = \mathcal{L}(\bar{x}'_{h_1}, \bar{x}_{-h_1}, \lambda)$. To see this note that:

$$\int_{\bar{x}'_{h_1}}^{\bar{x}_{h_1}} x - \frac{1-\lambda_{h_1}-F(x)}{f(x)} dF(x) + \int_{\bar{x}_{h_1}}^{1} \bar{x}_{h_1} - \frac{\lambda_{h_1}\bar{x}_{h_1}}{1-F(\bar{x}_{h_1})} dF(x) = \int_{\bar{x}'_{h_1}}^{1} \bar{x}'_{h_1} - \frac{\lambda_{h_1}\bar{x}'_{h_1}}{1-F(\bar{x}'_{h_1})} dF(x). \tag{47}$$

Equation (47) implies that $\bar{x}'_{h_1} - \frac{\lambda_{h_1}\bar{x}'_{h_1}}{1-F(\bar{x}'_{h_1})} > \bar{x}'_{h_1} - \frac{1-\lambda_{h_1}-F(\bar{x}'_{h_1})}{f(\bar{x}'_{h_1})}$, i.e., $\gamma_{h_1}(\bar{x}'_{h_1}) > \gamma_{h_1}^-(\bar{x}'_{h_1})$. So, $\lambda_{h_1} < \frac{\left(1-F\left(\bar{x}'_{h_1}\right)\right)^2}{1-F(\bar{x}'_{h_1})+\bar{x}'_{h_1} f(\bar{x}'_{h_1}))}$. The last inequality together with $\gamma_{h_1}^-(\bar{x}'_{h_1}) = \max_{i \neq h_1} \gamma_i(x_i)$ implies that $\bar{x}'_{h_1} > \max_{i \neq h_1} \bar{x}_i$.

So to complete the proof of Part 2, let us show that changing the threshold of bidder h_1 from $\bar{x}_{h_1}$ to $\bar{x}'_{h_1}$ does not change optimal λ minimizing $\mathcal{L}(.)$. Note that because $\gamma(\bar{x}_{h_1}) > \gamma(\bar{x}'_{h_1}) > \max_{i \neq h_1} \gamma_i(x_i)$, both $\mathcal{L}(\bar{x}, \lambda)$ and $\mathcal{L}(\bar{x}'_{h_1}, \bar{x}_{-h_1}, \lambda)$ are differentiable with respect to λ_{h_1}. Then (47) implies that $\frac{\partial \mathcal{L}(\bar{x},\lambda)}{\partial \lambda_{h_1}} = \frac{\partial \mathcal{L}(\bar{x}'_{h_1},\bar{x}_{-h_1},\lambda)}{\partial \lambda_{h_1}}$. Furthermore, for all $x \in [0,1]^n$, $\arg\max_{i\in\{1,...,n\}} \gamma_i(x_i)$ is the same under $(\bar{x},\lambda)$ and $(\bar{x}'_{h_1},\bar{x}_{-h_1},\lambda)$. Therefore, for all $h \in \mathbf{R}^n$, $\frac{\partial \mathcal{L}(\bar{x},\lambda+\epsilon h)}{\partial \epsilon}_{|\epsilon=0} = \frac{\partial \mathcal{L}(\bar{x}'_{h_1},\bar{x}_{-h_1},\lambda+\epsilon h)}{\partial \epsilon}_{|\epsilon=0}$. Since $\mathcal{L}(\bar{x},\lambda)$ is convex in λ for all $\bar{x}$, as shown in the proof of Lemma 11, the optimality of λ under $\bar{x}$ implies that $\frac{\partial \mathcal{L}(\bar{x},\lambda+\epsilon h)}{\partial \epsilon}_{|\epsilon=0} \geq 0$ for all $h \in \mathbf{R}^n$. Hence, $\frac{\partial \mathcal{L}(\bar{x}'_{h_1},\bar{x}_{-h_1},\lambda+\epsilon h)}{\partial \epsilon}_{|\epsilon=0} \geq 0$ for all $h \in \mathbf{R}^n$. The latter and convexity of $\mathcal{L}(\bar{x}'_{h_1},\bar{x}_{-h_1},\lambda)$ in λ implies that λ is the global minimum of $\mathcal{L}(\bar{x}'_{h_1},\bar{x}_{-h_1}.)$, as required. *Q.E.D.*

Lemma 5 has the following Corollary:

Corollary 1 *Equations (16) and (17) in Lemma 5 define a bijection between the set of thresholds* $(\bar{x}_1,...,\bar{x}_n)$ *and the set of Lagrange multipliers* $(\lambda_1,...,\lambda_n)$.

Proof of Corollary 1:

According to equation (17) the highest threshold $\bar{x}_{h_1} = \max_i \bar{x}_i$ in a solution to problem (14) satisfies $\bar{x}_{h_1} - \frac{1-F(\bar{x}_{h_1})}{f(\bar{x}_{h_1})} \leq \max_{j\neq h_1} \frac{\bar{x}_j^2 f(\bar{x}_j)}{1-F(\bar{x}_j)+\bar{x}_j f(\bar{x}_j)}$. This inequality provides an upper bound on $\bar{x}_{h_1}$, which is strictly less than 1 if $\max_{j:j\neq h_1} \bar{x}_j < 1$. Then, given $\bar{x} = (\bar{x}_1,...,\bar{x}_n)$ satisfying this upper bound on the highest threshold, equations (16) and (17) obviously define the profile $(\lambda_1,...,\lambda_n)$ uniquely. Let $\lambda(\bar{x})$ denote this profile.

To establish that a profile $\lambda = (\lambda_1,...,\lambda_n)$ uniquely defines a profile of thresholds via (16) and (17), note that $\lambda_i \leq 1-F(\bar{x}_i) \leq 1$ for all i by (16) and (17). Then, fix $(\lambda_1,...,\lambda_n) \in [0,1]^n$. For any i s.t. $\lambda_i \geq \lambda_j$ for some j, there is a unique solution $\bar{x}_i \in [0,1]$ to (16) because the right-hand side of (16) is decreasing in $\bar{x}_i$ and is equal to 1 (0) if $\bar{x}_i = 0$ ($\bar{x}_i = 1$).

Finally, if $(\lambda_1,...,\lambda_n)$ is such that $\lambda_{h_1} < \lambda_j$ for some h_1 and all $j \neq h_1$, then $\bar{x}_{h_1}$ is a solution to (17) which is well-defined because the right-hand side of (17) depends only on $\min_{j\neq h_1} \lambda_j$ and belongs to $[0,1]$ and the left-hand side of (17) (LHS) has the following three properties (a)-(c): (a) LHS is non-positive when $\bar{x}_{h_1} = 0$. (b) LHS is greater than the right-hand side of (17) when $\bar{x}_{h_1}$ is such that $F(\bar{x}_{h_1}) = 1 - \lambda_{h_1}$. To see this note that in this case the left-hand side of (17) is at least $\bar{x}_{h_1}$, the right hand side of it is no greater than $\max_{j\neq h_1} \bar{x}_j$, and $\bar{x}_{h_1} > \bar{x}_j$ for all j, $j \neq h_1$, because $1 - F(\bar{x}_{h_1}) = \lambda_{h_1} < \lambda_j \leq 1 - F(\bar{x}_j)$. The last inequality is satisfied because equation (16) holds for all j, $j \neq h_1$. (c) LHS is increasing in $\bar{x}_{h_1}$ when $F(\bar{x}_{h_1}) \leq 1 - \lambda_{h_1}$ by increasing hazard rate property. *Q.E.D.*

Proof of Lemma 6: Since $\gamma_i'(x_i) = 2 - f'(x_i)\frac{1-\lambda-F(x_i)}{f^2(x_i)}$ for $x_i \in [0,\bar{x}_i)$ and $\lambda_i \leq 1-F(\bar{x}_i)$ for all i by Lemma 5, it is immediate that $\gamma_i'(x_i) > 0$ if $f'(x_i) \geq 0$. On the other hand, if $f'(x_i) < 0$, then $\gamma_i'(x_i) \geq \left(x_i - \frac{1-F(x_i)}{f(x_i)}\right)' > 0$, where the first inequality holds because $\lambda_i \geq 0$, and the second inequality holds by the increasing hazard rate property. So, $\gamma_i(x_i)$ is strictly increasing on $[0,\bar{x}_i)$. Then by Lemma 5, $\gamma_i(\bar{x}_i) > \gamma_i(x_i)$ for all $x_i \in [0,\bar{x}_i)$.

Lemma 4 then implies that $Q_i(x_i, x_{-i})$ and hence $q_i(x_i)$ are both increasing in x_i. So a solution to the relaxed problem is also a solution to the full problem. *Q.E.D.*

Proof of Lemma 7: Since $q_i(x_i) \equiv \int_{x_{-i}\in[0,1]^{n-1}} Q_i(x_i, x_{-i}) \prod_{j\neq i} dF(x_j)$, the inequalities in (18) immediately follow from Lemma 4.

Let $Z_i(x_i) = \{x_{-i}|\gamma_i(x_i) = \gamma_j(x_j) \text{ for some } j \neq i\}$. By Lemma 6, $\gamma_i(x_i)$ is strictly increasing on $[0, \bar{x}_i)$ for all i. So for almost all $x_i \in [0, \bar{x}_i)$, $Z_i(x_i)$ is at most finite, and hence the left-hand and right-hand sides of (18) are equal, and so $q_i(x_i)$ is uniquely defined by (18).

Now, consider $\bar{x}_i$. If $\bar{x}_i \neq \bar{x}_j$ for all $j \neq i$, then the set $Z_i(\bar{x}_i)$ has measure zero. Hence, the left- and right-hand sides in (18) are equal, and $q_i(\bar{x}_i)$ is uniquely defined by (18).

The left-and the right-hand sides of (18) depend only on x_i and the profile $(\bar{x}_1, ..., \bar{x}_n, \lambda_1, ..., \lambda_n)$. But by Corollary 1 equations (16) and (17) define a bijection between $(\bar{x}_1, ..., \bar{x}_n)$ and $(\lambda_1, ..., \lambda_n)$ in an optimal mechanism. So $q_i(x_i)$ is determined by the profile $(\bar{x}_1, ..., \bar{x}_n)$, or equivalently, by the profile $(\lambda_1, ..., \lambda_n)$ when the left- and the right-hand sides of (18) are equal, which is true a.e. on $[0, \bar{x}_i]$ and for all $\bar{x}_i$ s.t. $\bar{x}_i \neq \bar{x}_j$ for all $j \neq i$. *Q.E.D.*

Proof of Lemma 8: To argue by contradiction, suppose that $\bar{x}_j > \bar{x}_i$. Then by part 1 of Lemma 5, $\lambda_i > 0$. Hence, i's budget constraint is binding, i.e., $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s)ds$.

Since $\bar{x}_j > \bar{x}_i$, by Lemma 5 $\lambda_j < \lambda_i \leq 1 - F(\bar{x}_i)$ and so $\gamma_j(\bar{x}_j) > \gamma_i(\bar{x}_i)$ and $\gamma_j(x) < \gamma_i(x)$ for all $x \in [0, \bar{x}_i]$. So by Lemma 4, $q_j(\bar{x}_j) \geq q_i(\bar{x}_i)$ and $q_j(x) \leq q_i(x)$ for all $x \in [0, \bar{x}_i]$. Also, $q_j(.)$ is nondecreasing by Lemma 6. Therefore, we have:

$$m_j \geq \bar{x}_j q_j(\bar{x}_j) - \int_0^{\bar{x}_j} q_j(s)ds = \bar{x}_i q_j(\bar{x}_j) + \int_{\bar{x}_i}^{\bar{x}_j} (q_j(\bar{x}_j) - q_j(s))ds - \int_0^{\bar{x}_i} q_j(s)ds \geq$$

$$\bar{x}_i q_j(\bar{x}_j) - \int_0^{\bar{x}_i} q_j(s)ds \geq \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s)ds = m_i \tag{48}$$

But (48) contradicts $m_i > m_j$. Hence, we must have $\bar{x}_i \geq \bar{x}_j$. *Q.E.D.*

Proof of Lemma 9: First, suppose that $m^1 \leq p^m = \arg\max_p p(1 - F(p))$. Note that $p^m - \frac{1-F(p^m)}{f(p^m)} = 0$. Let us show that bidder 1's budget constraint is binding in an optimal mechanism. The proof is by contradiction, so suppose not, i.e., $m_1 > \bar{x}_1 q_1(\bar{x}_1) - \int_0^{\bar{x}_1} q_1(x)dx$. Then $\lambda_1 = 0$, and $\gamma_1(x_1) = x_1 - \frac{1-F(x_1)}{f(x_1)}$ for $x_1 < \bar{x}_1$. So, $q_1(x_1) = 0$ for $x_1 < \min\{p^m, \bar{x}_1\}$.

We need to consider two cases: $\bar{x}_1 = 1$ and $\bar{x}_1 < 1$. First, suppose that $\bar{x}_1 = 1$. Then $\gamma_1(\bar{x}_1) = 1$. By Lemma 5, $\gamma_i(x_i) < 1$ for all i and $x_i < 1$. So, $q_1(1) = 1$ by Lemma 7.

Now, suppose that $\bar{x}_1 < 1$. Since $\lambda_1 = 0$, by Lemma 5 we must have $\bar{x}_1 > \bar{x}_i$ for all $i \neq 1$ and $\gamma_1(\bar{x}_1) > \gamma_1^-(\bar{x}_1) = \max_{i\neq 1, x_i\in[0,1]} \gamma_i(x_i)$. So $\gamma_1^-(\bar{x}_1) > 0$ and hence $\bar{x}_1 \geq p^m$ by equation (17). Also, by Lemma 7, $q_1(\bar{x}_1) = 1$. So, for both $\bar{x} = 1$ and $\bar{x}_1 < 1$, we have:

$$m_1 > \bar{x}_1 q_1(\bar{x}_1) - \int_0^{\bar{x}_1} q_1(x_1)dx_1 = \bar{x}_1 - \int_{p^m}^{\bar{x}_1} q_1(x_1)dx_1 \geq 1 - \int_{p^m}^{1} 1dx_1 = p_m,$$

which contradicts the assumption that $m_1 \le p^m$.

Now, consider a bidder $i \neq 1$. By Lemma 8 $\bar{x}_i \le \bar{x}_1$. If $\bar{x}_i < 1$, then $\lambda_i > 0$ by (16), and so the budget constraint of i is binding. Now suppose that $\bar{x}_i = 1$. If the budget constraint of i is not binding, then $\lambda_i = 0$. However, with $\bar{x}_i = 1$ and $\lambda_i = 0$, the same argument as for bidder 1 in case $\bar{x}_1 = 1$ shows that $q_i(1) = 1$ and $q_i(x_i) = 0$ for $x_i < p^m$. But then $m_i > \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(x_i)dx_i \ge 1 - \int_{p^m}^1 1dx_1 = p_m$. A contradiction. *Q.E.D.*

Proof of Lemma 10: The strong duality property holds and (x^*, λ^*) is the solution to both the primal problem $\max_x \min_\lambda \mathcal{L}(\bar{x}, \lambda)$ and its dual, $\min_\lambda \max_x \mathcal{L}(\bar{x}, \lambda)$, if and only if (x^*, λ^*) is a saddle point of $\mathcal{L}(\bar{x}, \lambda)$ (see e.g. Prop. 1.3.7, Ch. 1 in Bertsekas (2001)),i.e.,

$$\mathcal{L}(\bar{x}, \lambda^*) \le \mathcal{L}(x^*, \lambda^*) \le \mathcal{L}(x^*, \lambda) \text{ for all } \bar{x} \in [0,1]^n \text{ and } \lambda \in [0,1]^n. \tag{49}$$

So, to complete the proof we will establish the existence of (x^*, λ^*) s.t. (49) holds.

Consider dual function $g(\lambda) \equiv \max_{\bar{x}\in[0,1]^n} \mathcal{L}(\bar{x}, \lambda) = \mathcal{L}(\bar{x}(\lambda), \lambda)$, where $\bar{x}(\lambda)$ is defined by (16) and (17) in Lemma 5. Since $\mathcal{L}(\bar{x}, \lambda)$ is continuous in $(\bar{x}, \lambda)$, by Berge's Maximum Theorem $g(\lambda)$ is well-defined and continuous and, therefore, possesses a minimizer, λ^*.

Below, we will establish that this minimizer is unique. However, let us first show that (49) holds for the pair (x^*, λ^*) where $x^* = \bar{x}(\lambda^*)$. First, since $x^* = \bar{x}(\lambda^*)$, $\mathcal{L}(\bar{x}, \lambda^*) \le \mathcal{L}(x^*, \lambda^*)$ for all $\bar{x} \in [0,1]^n$ by Lemma 5, i.e., the first inequality in (49) holds.

To show that $\mathcal{L}(x^*, \lambda^*) \le \mathcal{L}(x^*, \lambda)$ for all λ, recall that, as shown in the proof of Lemma 5, $\mathcal{L}(\bar{x}, \lambda)$ is convex in λ for all $\bar{x}$. So, by Danskin's Theorem (Bertsekas (2001), p. 131), $g(\lambda)$ is convex and $\frac{\partial \mathcal{L}(x^*, \lambda^* + \epsilon h)}{\partial \epsilon}\Big|_{\epsilon=0} = \frac{\partial g(\lambda^* + \epsilon h)}{\partial \epsilon}\Big|_{\epsilon=0}$ for all $h \in \mathbf{R}^n$. But, $\frac{\partial g(\lambda^* + \epsilon h)}{\partial \epsilon}\Big|_{\epsilon=0} \ge 0$ since $\lambda^* \in \arg\min_\lambda g(\lambda)$. So, $\frac{\partial \mathcal{L}(x^*, \lambda^* + \epsilon h)}{\partial \epsilon}\Big|_{\epsilon=0} \ge 0$ for all $h \in \mathbf{R}^n$ and, since $\mathcal{L}(x^*, \lambda)$ is convex in λ, λ^* is its global minimum. This completes the proof that (x^*, λ^*) is a saddle point of $\mathcal{L}(\bar{x}, \lambda)$.

To show that the solution to the dual problem is unique, it is sufficient to prove that $g(\lambda)$ has a unique minimum. The latter is true because $g(\lambda)$ is strictly convex, i.e., for all $\lambda^1, \lambda^2 \in [0,1]^n$, $\lambda^1 \neq \lambda^2$, and $\alpha \in (0,1)$, $g(\alpha\lambda^1 + (1-\alpha)\lambda^2) < \alpha g(\lambda^1) + (1-\alpha)g(\lambda^2)$. Indeed, we have: $g(\alpha\lambda^1 + (1-\alpha)\lambda^2) = \mathcal{L}(\bar{x}(\alpha\lambda^1 + (1-\alpha)\lambda^2), \alpha\lambda^1 + (1-\alpha)\lambda^2)) \le \alpha\mathcal{L}(\bar{x}(\alpha\lambda^1 + (1-\alpha)\lambda^2), \lambda^1) + (1-\alpha)\mathcal{L}(\bar{x}(\alpha\lambda^1 + (1-\alpha)\lambda^2), \lambda^2) < \alpha\mathcal{L}(\bar{x}(\lambda^1), \lambda^1) + (1-\alpha)\mathcal{L}(\bar{x}(\lambda^2), \lambda^2) = \alpha g(\lambda^1) + (1-\alpha)g(\lambda^2)$. Note that the first inequality holds by convexity of $\mathcal{L}(\bar{x}, \lambda)$ in λ, and the second inequality holds because by Lemma 5 and its Corollary 1, the optimal threshold profile $\bar{x}(.)$ is such that $\bar{x}(\lambda) \neq \bar{x}(\lambda')$ when $\lambda \neq \lambda'$. *Q.E.D.*

Proof of Theorem 1: By Lemma 10, the optimal mechanism can be computed by minimizing strictly convex and continuous dual function $g(\lambda) \equiv \mathcal{L}(\bar{x}(\lambda), \lambda)$ on $[0,1]^n$. Strict convexity of $g(.)$ implies that $g(.)$ has a unique minimum λ satisfying the first-order conditions

$g'(\lambda; h) \geq 0$ for any $h \in \mathbf{R}^2$, where $g'(\lambda; h)$ is the directional derivative of $g(\lambda)$ in the direction h. Since $\bar{x}(\lambda)$ is unique, by Danskin's Theorem $g'(\lambda; h) = \frac{\partial \mathcal{L}(\lambda+\epsilon h, \bar{x}(\lambda))}{\partial \epsilon}_{|\epsilon=0+}$.

In this Theorem we deal with the case $\lambda_1 = \lambda_2 = \lambda^t$, when this set of the first-order conditions is equivalent to the following: (i) $\frac{\partial_+ g(\lambda_i,\lambda_j)}{\partial \lambda_i}_{|\lambda_i=\lambda_j=\lambda^t} \geq 0$, (ii) $\frac{\partial_- g(\lambda_i,\lambda_j)}{\partial \lambda_i}_{|\lambda_i=\lambda_j=\lambda^t} \leq 0$ for $i \in \{1,2\}$, $j \neq i$, where ∂_+ (∂_-) denotes the right-hand (left-hand) derivative; (iii) $\frac{\partial g(\lambda_1+\epsilon,\lambda_2+\epsilon)}{\partial \epsilon}_{|\epsilon=0+,\lambda_1=\lambda_2=\lambda^t} \geq 0$; (iv) $\frac{\partial g(\lambda_1+\epsilon,\lambda_2+\epsilon)}{\partial \epsilon}_{|\epsilon=0-,\lambda_1=\lambda_2=\lambda^t} \leq 0$. Conditions (i) and (ii) involve variations in one of the λ's; (iii) and (iv) involve simultaneous variations in both λ's.

Starting from the first-order condition (i) and differentiating (19) we may compute:

$$
\begin{aligned}
&\frac{\partial_+ g(\lambda_i,\lambda_j)}{\partial \lambda_i}_{|\lambda_i=\lambda_j=\lambda^t} = \frac{\partial_+ \mathcal{L}(\lambda_i,\lambda_j,\bar{x}(\lambda_i,\lambda_j))}{\partial \lambda_i}_{|\lambda_i=\lambda_j=\lambda^t} = \\
&m_i + \int_{(x_i,x_j)\in[0,1]:\max\{\gamma_i(x_i),\gamma_j(x_j)\}\geq 0} \frac{\partial_+ \max\{\gamma_i(x_i),\gamma_j(x_j)\}}{\partial \lambda_i}_{|\lambda_i=\lambda_j=\lambda^t} dF(x_j)dF(x_i) = \\
&m_i + \int_{\{(x_i,x_j)\in[\bar{x}^t,1]\times[0,\bar{x}^t)\}\cup\{(x_i,x_j)\in[0,\bar{x}^t)\cup[0,x_i]:\ \gamma_i(x_i)\geq 0\}} \frac{\partial \gamma_i(x_i)}{\partial \lambda_i} dF(x_j)dF(x_i) \\
&= m_i - \bar{x}^t \int_{x_j\in[0,\bar{x}^t)} dF(x_j) + \int_{(x_i,x_j)\in[0,\bar{x}^t)\times[0,x_i]:\gamma_i(x_i)\geq 0} dF(x_j)dx_i \\
&= m_i - \bar{x}^t F(\bar{x}^t) + \int_{x_i:\gamma_i(x_i)\geq 0}^{\bar{x}^t} F(x_i)dx_i \geq 0. \qquad (50)
\end{aligned}
$$

The third equality holds because $\gamma_i(x) = \gamma_j(x)$ for all x, $\gamma_i(.)$ is strictly increasing on $[0, \bar{x}^t]$, with $\gamma_i(\bar{x}_i) \geq 0$, $\frac{\partial \gamma_i(\bar{x}^t)}{\partial \lambda_i} = -\frac{\bar{x}^t}{1-F(\bar{x}^t)} \leq 0$, $\frac{\partial \gamma_i(x_i)}{\partial \lambda_i} = \frac{1}{f(x_i)} > 0$ if $x_i < \bar{x}^t$. So, $\frac{\partial_+ \max\{\gamma_i(x_i),\gamma_j(x_j)\}}{\partial \lambda_i}_{|\lambda_i=\lambda_j=\lambda^t}$ is equal to: (a) $\frac{\partial \gamma_i(x_i)}{\partial \lambda_i}_{|\lambda_i=\lambda^t}$ when either $x_i \in [\bar{x}^t, 1]$, $x_j \in [0, \bar{x}^t)$ or $x_j \leq x_i < \bar{x}^t$, $\gamma_i(x_i) \geq 0$; (b) zero otherwise. The fourth equality holds because $\frac{\partial \gamma_i(\bar{x}^t)}{\partial \lambda_i} = -\frac{\bar{x}^t}{1-F(\bar{x}^t)}$ and $\frac{\partial \gamma_i(x_i)}{\partial \lambda_i} = \frac{1}{f(x_i)}$ if $x_i < \bar{x}^t$. The fifth equality holds by integration.

Next, consider first-order condition (ii). Using steps similar to those in (50) yields:

$$
\frac{\partial_- g(\lambda_i,\lambda_j)}{\partial \lambda_i}_{|\lambda_i=\lambda_j=\lambda^t} = \frac{\partial_- \mathcal{L}(\lambda_i,\lambda_j,\bar{x}(\lambda_i,\lambda_j))}{\partial \lambda_i}_{|\lambda_i=\lambda_j=\lambda^t} = m_i - \bar{x}^t + \int_{x_i:\gamma_i(x_i)\geq 0}^{\bar{x}^t} F(x_i)dx_i \leq 0. \quad (51)
$$

From (50) it follows that $\lambda^t > 0$. To see this, suppose $\lambda^t = 0$. Then $\bar{x}^t = 1$ by Lemma 5, using which in (50) yields $m_2 - 1 + \int_{r:r-\frac{1-F(r)}{f(r)}=0}^{1} F(x)dx \geq 0$, contradicting $m_2 < \widehat{m}$.

Now consider first-order conditions (iii) and (iv). Despite the max operator in $g(.)$, it is differentiable w.r.t. ϵ in (iii) and (iv), because $\frac{\partial \max\{\gamma_i(x_i),\gamma_j(x_j)\}}{\partial \epsilon}_{|\epsilon=0,\lambda_i=\lambda_j=\lambda^t+\epsilon} = \frac{\partial \gamma_i(x_i)}{\partial \lambda_i}_{|\lambda_i=\lambda^t}$

when $\gamma_i(x_i) = \gamma_j(x_j)$. Since $\lambda^t > 0$, (iii) and (iv) must hold as equalities, i.e., we have:

$$\frac{\partial g(\lambda_1+\epsilon, \lambda_2+\epsilon)}{\partial \epsilon}_{|\epsilon=0, \lambda_1=\lambda_2=\lambda^t} = \frac{\partial \mathcal{L}(\bar{x}(\lambda^t, \lambda^t), \lambda^t+\epsilon, \lambda^t+\epsilon)}{\partial \epsilon}_{|\epsilon=0} =$$

$$\sum_{i\in\{1,2\}} m_i + \int_{(x_1,x_2)\in[0,1]^2:\, \max_{i\in\{1,2\}}\gamma_i(x_i)\geq 0} \frac{\partial \max_{i\in\{1,2\}}\gamma_i(x_i)}{\partial \epsilon}_{|\lambda_1=\lambda_2=\lambda^t+\epsilon, \epsilon=0} \prod_{i\in\{1,2\}} dF(x_i) =$$

$$\sum_{i=1,2}\left(m_i + \int_{(x_i,x_j)\in[0,\bar{x}^t)\times[0,x_i):\gamma_i(x_i)\geq 0} \frac{\partial \gamma_i(x_i)}{\partial \lambda_i}_{|\lambda_i=\lambda^t} dF(x_j)dF(x_i)\right) + (1+F(\bar{x}^t))\int_{\bar{x}^t}^1 \frac{\partial\gamma_i(x_i)}{\partial\lambda_i}dF(x_i)$$

$$= \sum_{i=1,2}\left(m_i + \int_{x_i\in[0,\bar{x}^t):\gamma_i(x_i)\geq 0} F(x_i)dx_i\right) - \bar{x}^t(1+F(\bar{x}^t)) = 0 \tag{52}$$

The second equality in (52) holds by definition. The third is obtained by breaking up the range of integration $\{x_1, x_2 : \max_{i\in\{1,2\}}\gamma_i(x_i) \geq 0\}$ into two regions: the first one where $\max\{x_1, x_2\} < \bar{x}^t$ and the second one where $\max\{x_1, x_2\} \geq \bar{x}^t$, and taking into account that $\gamma_i(x_i) > \max\{0, \gamma_j(x_j)\}$ when $x_i \geq \bar{x}^t > x_j$, and $\gamma_i(x_i) = \gamma_j(x_j)$ and $\frac{\partial\gamma_i(x_i)}{\partial\lambda_i} = \frac{\partial\gamma_j(x_j)}{\partial\lambda_j} =$ when $x_i, x_j \in [\bar{x}^t, 1]$. The fourth equality holds because: (i) by (12) $\frac{\partial\gamma_i(x_i)}{\partial\lambda_i} = \frac{1}{f(x_i)}$ if $x_i < \bar{x}^t$ and $\gamma_i(x_i) \geq \gamma_j(x_j)$ iff $x_i \geq x_j$; (ii) $\frac{\partial\gamma_i(x_i)}{\partial\lambda_i} = -\frac{\bar{x}^t}{1-F(\bar{x}^t)}$ if $x_i \geq \bar{x}^t$, so that $\int_{\bar{x}^t}^1 \frac{\partial\gamma_i(x_i)}{\partial\lambda_i}dF(x_i) = -\bar{x}^t$.

To summarize, (50)-(52) are the first-order conditions for the case $\lambda_1 = \lambda_2 = \lambda^t$. Since $g(\lambda)$ is strictly convex, its unique minimum must satisfy these first-order conditions. So, (50)-(52) are necessary and sufficient for $\lambda_1 = \lambda_2 = \lambda^t$ and hence, for $\bar{x}_1 = \bar{x}_2 = \bar{x}^t$.

Let us now show that (50)-(52) are equivalent to (20)-(21). First, (52) is equivalent to (20) since r_t in the statement of the Theorem is the unique solution to $\gamma_i(r_t) = 0$.

Since $m_1 \geq m_2$, (50) holds for $i \in \{1, 2\}$ if and only if it holds for $i = 2$. Likewise, (51) holds for $i \in \{1, 2\}$ if and only if it holds for $i = 1$. Further, subtracting (50) for $i = 2$ from (51) for $i = 1$ we obtain $m_1 - m_2 \leq \bar{x}^t(1 - F(\bar{x}^t))$, which is (21).

On the other hand, suppose that (20)-(21) hold. Then adding (20) and (21) we obtain (51) for $i = 1$, while subtracting (21) from (20) yields (50) for $i = 2$.

Further, by Lemma 6 $\gamma_i(x)$ is strictly increasing on $[0, \bar{x}_i]$. So, since $\gamma_1(x) = \gamma_2(x)$, by Lemma 4, $q_1(x) = q_2(x) = F(x)$ for all $x \in [r_t, \bar{x}^t)$ and $q_1(x) = q_2(x) = 0$ for all $x \in [0, r_t)$.

Finally, since $\gamma_1(\bar{x}^t) = \gamma_2(\bar{x}^t) > \gamma_1(x) = \gamma_2(x)$ for all $x \in [0, x^t)$, by Lemma 4 $q_1(\bar{x}^t) + q_1(\bar{x}^t) = 1 + F(\bar{x}^t)$. The latter and (20) imply that budget constraints (22) must be binding for $i \in \{1, 2\}$. These constraints uniquely define $q_1(\bar{x}^t)$ and $q_2(\bar{x}^t)$. Since $m_1 \geq m_2$, $q_1(\bar{x}^t) \geq q_2(\bar{x}^t)$. Also, combining (20)-(22) yields $q_1(\bar{x}^t) \leq 1$ and $q_2(\bar{x}^t) \geq F(\bar{x}^t)$. *Q.E.D.*

Proof of Theorem 2: Recall that the optimal mechanism is characterized by the unique minimum (λ_1, λ_2) of the strictly convex dual function $g(.)$. Since (20)-(21) do not hold, we must have $\lambda_1 \neq \lambda_2$. Since $m_1 > m_2$, by Lemma 5 and Lemma 8 we have $\lambda_1 < \lambda_2$,

$\bar{x}_1 > \bar{x}_2$, and $\gamma_1(\bar{x}_1) > \gamma_2(\bar{x}_2)$. So the ties when $\gamma_1(x_1) = \gamma_2(x_2)$ occur with zero probability. Therefore, $g(\lambda)$ is differentiable and its minimum (λ_1, λ_2) is a unique solution to the first-order conditions $\frac{\partial g(\lambda)}{\partial \lambda_i} = 0$ if $\lambda_i > 0$ and $\frac{\partial g(\lambda)}{\partial \lambda_i} \geq 0$ if $\lambda_i = 0$, $i \in \{1, 2\}$. Since the optimal $\bar{x}(\lambda)$ is unique, by Danskin Theorem $\frac{\partial g(\lambda)}{\partial \lambda_i} = \frac{\partial \mathcal{L}(\bar{x}(\lambda),\lambda)}{\partial \lambda_i}$. Then differentiating (19) yields:

$$\frac{\partial g(\lambda)}{\partial \lambda_i} = \frac{\partial \mathcal{L}(\bar{x}(\lambda), \lambda)}{\partial \lambda_i} = m_i + \int_{(x_i,x_j):\gamma_i(x_i)>\max\{0,\gamma_j(x_j)\}} \frac{\partial \gamma_i(x_i)}{\partial \lambda_i} dF(x_i)dF(x_j) = m_i -$$
$$\int_{x_i\in[\bar{x}_i,1],x_j\in[0,1]:\gamma_i(\bar{x}_i)>\gamma_j(x_j)} \frac{\bar{x}_i}{1 - F(\bar{x}_i)} dF(x_j)dF(x_i) + \int_{x_i\in[0,\bar{x}_i),x_j\in[0,1]:\gamma_i(x_i)>\max\{0,\gamma_j(x_j)\}} dF(x_j)dx_i$$
$$= m_i - \bar{x}_i q_i(\bar{x}_i) + \int_0^{\bar{x}_i} q_i(x_i)dx_i. \tag{53}$$

The second equality in (53) holds by definition. The third holds because by (12) we have: $\frac{\partial \gamma_i(x_i)}{\partial \lambda_i} = \frac{1}{f(x_i)}$ if $x_i < \bar{x}_i$ and $\frac{\partial \gamma_i(x_i)}{\partial \lambda_i} = -\frac{\bar{x}_i}{1-F(\bar{x}_i)}$ if $x_i \geq \bar{x}_i$. The fourth holds by Lemma 7.

By (53) $\frac{\partial g(\lambda)}{\partial \lambda_i}_{|\lambda_i \geq 1} > 0$. Indeed, $\lambda_i \geq 1$ implies that $\bar{x}_i q_i(\bar{x}_i) = 0$ because in this case $\gamma_i(\bar{x}_i) \leq 0$, with equality only if $\bar{x}_i = 0$. So we must have $\lambda_i \in [0, 1)$.

Since $\bar{x}_1 > \bar{x}_2$, Lemma 5 implies that $\lambda_1 < \lambda_2$ and $\gamma_1(\bar{x}_1) > \gamma_1^-(\bar{x}_1) = \gamma_2^-(\bar{x}_2) = \gamma_2(\bar{x}_2)$. So, $\lambda_2 > 0$, and by Lemma 4 $q_1(\bar{x}_1) = 1$ and $q_2(\bar{x}_2) = F(\bar{x}_1)$.

Since $\lambda_1 < \lambda_2$, by monotone hazard rate $\gamma_1(x) < \gamma_2(x)$ for $x \in [0, \bar{x}_2]$. Also, $\gamma_1(x) < \gamma_2(\bar{x}_2)$ for all $x \in (\bar{x}_2, \bar{x}_1)$ since $\gamma_1^-(\bar{x}_1) = \gamma_2^-(\bar{x}_2)$. So, by Lemma 7 $q_1(x) < F(x)$ for all $x \in [r_1, \bar{x}_1)$, $q_2(x) > F(x)$ for all $x \in [r_2, \bar{x}_2)$, where r_i solves $\gamma_i(r_i) = r_i - \frac{1-F(r_i)-\lambda_i}{f(r_i)} = 0$, and so $r_1 > r_2$.

Next, consider two cases. Case 1: Bidder 1's budget constraint is binding, so (53) is zero for $i = 1$. Case 2. Bidder 1's budget constraint is non-binding, so (53) is positive for $i = 1$.

Let us start from Case 1. Then $\lambda_1 = 0$ and by Lemma 5, $\lambda_2 = \frac{(1-F(\bar{x}_2))^2}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}$. So equation (26) follows from (17). To see that (25) and (26) determine $\bar{x}_1$ and $\bar{x}_2$, note that by Lemma 7, $q_i(x_i) = 0$ for all $x_i \in [0, r_i)$ and $q_i(x_i) = F(\psi^j(x_i|\bar{x}_2))$ for all $x_i \in [r_i, \bar{x}_i)$, $i, j \in \{1, 2\}$, $i \neq j$, where $\psi^2(x_1|\bar{x}_2)$ ($\psi^1(x_2|\bar{x}_2)$) is the unique solution for x_2 (x_1) to the equation $\gamma_1(x_1) = \gamma_2(x_2)$ which by (12) and (16) we can rewrite as follows:

$$x_1 - \frac{1 - F(x_1)}{f(x_1)} = x_2 - \frac{1 - F(x_2) - \frac{(1-F(\bar{x}_2))^2}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}}{f(x_2)}. \tag{54}$$

Note that $\psi^i(x_j|\bar{x}_2)$ is increasing in x_j, $\psi^i(\bar{x}_j|\bar{x}_2) = \bar{x}_i$ by (26), and we can rewrite (25) as:

$$m_2 = \bar{x}_2 F(\psi^1(\bar{x}_2|\bar{x}_2)) - \int_{r_2}^{\bar{x}_2} F(\psi^1(x_2|\bar{x}_2))dx_2. \tag{55}$$

The right-hand side of (55) is continuous in $\bar{x}_2$ because $\psi^1(.|.)$ is continuous, is equal to $\widehat{m} = 1 - \int_{p^m}^1 F(x)dx > m_2$ when $\bar{x}_2 = 1$ and is equal to zero when $\bar{x}_2 = 0$. Furthermore, the derivative of the right-hand side of (55) with respect to $\bar{x}_2$ is equal to:

$$\bar{x}_2 f(\psi^1(\bar{x}_2|\bar{x}_2)) \frac{d\psi^1(\bar{x}_2|\bar{x}_2)}{d\bar{x}_2} - \int_{r_2}^{\bar{x}_2} f(\psi^1(x_2|\bar{x}_2)) \frac{\partial \psi^1(x_2|\bar{x}_2)}{\partial \bar{x}_2} dx_2 + F(\psi^1(r_2|\bar{x}_2)) \frac{dr_2}{d\bar{x}_2} > 0.$$

This derivative is positive since each of its terms is positive. So, by the inverse function theorem, there exists a unique solution $\bar{x}_2(m_2) \in (0,1)$ to (55), and it is continuous and strictly increasing in m_2.

Now consider the corresponding threshold of bidder 1, $\bar{x}_1(m_2) = \psi^1(\bar{x}_2(m_2)|\bar{x}_2(m_2))$. By (54), $1 > \bar{x}_1(m_2) > \bar{x}_2(m_2)$, and the transfer that type $\bar{x}_1(m_2)$ pays in this mechanism is:

$$\tilde{m}(m_2) \equiv \bar{x}_1(m_2) - \int_{r_1}^{\bar{x}_1(m_2)} F(\psi^2(x_1|\bar{x}_2(m_2))dx_1. \tag{56}$$

So, if $m_1 > \tilde{m}(m_2)$, i.e., (27) holds, then the budget constraint of bidder 1 is non-binding and the mechanism that we have constructed is optimal. Note that $\tilde{m}(m_2)$ is uniquely determined by m_2 and continuous in it because, as shown above, $\bar{x}_1(m_2)$ and $\bar{x}_2(m_2)$ are uniquely determined by m_2 and continuous. Comparing (55) and (56) observe that $\tilde{m}(m_2) > m_2$ because $\bar{x}_1(m_2) > \bar{x}_2(m_2)$, $r_1 > r_2$ and $\psi^2(x_1|\bar{x}_2) < x_1$, while $\psi^1(x_2|\bar{x}_2) > x_2$.

Now suppose that $m_1 \leq \tilde{m}(m_2)$, i.e., (27) fails. Then we must be in Case 2 where $\frac{\partial g(\lambda)}{\partial \lambda_i} = m_i - \bar{x}_i q_i(\bar{x}_i) + \int_0^{\bar{x}_i} q_i(x_i)dx_i = 0$, $i \in \{1,2\}$, and $\bar{x}_1$ and $\bar{x}_2$ are determined by these two budget constraints. The existence and uniqueness of such $\bar{x}_1$ and $\bar{x}_2$ follows from the existence and the uniqueness of the minimum of $g(\lambda)$ and the fact that this minimum must be the unique solution to the first-order conditions $\frac{\partial g(\lambda)}{\partial \lambda_i} = 0$, $i \in \{1,2\}$.

As in Case 1, $q_i(x_i)$, $i \in \{1,2\}$ are determined by $\bar{x}_1$ and $\bar{x}_2$ according to Lemma 7 Specifically, $q_i(x_i) = 0$ for $x_i \in [0, r_i)$; $q_i(x_i) = F(\phi^j(x_i|\bar{x}_1, \bar{x}_2))$ for $x_i \in [r_i, \bar{x}_i)$, $i, j \in \{1,2\}$, $i \neq j$, where $\phi^2(x_1|\bar{x}_1, \bar{x}_2)$ ($\phi^1(x_2|\bar{x}_1, \bar{x}_2)$) is the unique solution for x_2 (x_1) to the equation $\gamma_1(x_1) = \gamma_2(x_2)$, or equivalently, $x_1 - \frac{1-F(x_1)-\lambda_1}{f(x_1)} = x_2 - \frac{1-F(x_2)-\lambda_2}{f(x_2)}$, where by Lemma 5: $\lambda_1 = 1 - F(\bar{x}_1) - f(\bar{x}_1)\left(\bar{x}_1 - \frac{\bar{x}_2^2 f(\bar{x}_2)}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}\right)$ and $\lambda_2 = \frac{(1-F(\bar{x}_2))^2}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}$. *Q.E.D.*

Proof of Theorem 3: (i) Since the bidders' valuations are identically distributed, $\pi(.)$ is exchangeable, i.e., $\pi(m_1, m_2) = \pi(m_2, m_1)$. For (m_1, m_2) such that $\sum_i m_i = M$, by concavity of $\pi(.)$ we have $\pi\left(\frac{M}{2}, \frac{M}{2}\right) \geq \frac{\pi(m_1,m_2)+\pi(m_2,m_1)}{2} = \pi(m_1, m_2)$. So, $\pi^*(M) = \pi\left(\frac{M}{2}, \frac{M}{2}\right)$.

Next, suppose that (m_1, m_2) is such that $m_1 + m_2 = M$ and (20)-(21) hold. Then the optimal mechanism is a top auction with the same threshold $\bar{x}^t$ as under $\left(\frac{M}{2}, \frac{M}{2}\right)$, since $\bar{x}^t$ depends only on M. By Lemma 5, the Lagrange multiplier of each bidder under both budget profiles is $\lambda^t = \frac{\left(1-F\left(\bar{x}^t\right)\right)^2}{(1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t))}$ and hence, by (29), $\pi(m_1, m_2) = \pi\left(\frac{M}{2}, \frac{M}{2}\right)$.

(ii) Now consider a budget profile (m_1', m_2') s.t. $m_1' + m_2' = M$ and $m_1 - m_2 > 0$ is sufficiently large that (20)- (21) fail, so the optimal mechanism is a budget handicap auction. Let λ_1' and λ_2' be the optimal values of Lagrange multipliers under (m_1', m_2'). By Lemma 5, $\lambda_1' < \lambda_2'$. Then from (29) it follows that $\frac{\partial \pi(m_1'-\epsilon, m_2'+\epsilon)}{\partial \epsilon}_{|\epsilon=0} = -\lambda_1' + \lambda_2' > 0$, and so $\pi(m_1', m_2') < \pi(m_1, m_2)$ for any (m_1, m_2) s.t. $m_1 + m_2 = M$ and $m_2 > m_2'$. *Q.E.D.*

Online Appendix B (Not for Publication)

This Appendix consists of several sections. Section 1B contains the formal results for $n > 2$ bidder case. In section 2B we present examples with 2 and 3 bidders and uniform type distribution. Section 3B deals with constrained-efficient mechanism. Section 4B presents a generalized top auction with asymmetric type distribution.

Section 1B. Top and Budget-Handicap Auctions with n Bidders

In this section we provide a formal characterization of the optimal mechanism for n bidders. The optimal mechanism also takes the form of a top auction or a budget-handicap auction, similar to those in the 2-bidder case considered in Section 4. The proofs of these results are analogous to their counterparts for 2 bidders and are therefore omitted. In particular, the same argument as in Section 4 establishes that our case of interest is when the lowest budget m_n satisfies $m_n < \widehat{m}(n) \equiv 1 - \int_{r:r-\frac{1-F(r)}{f(r)}=0}^{1} F(x)^{n-1}dx$, since otherwise the optimal mechanism is the same as without budget constraints.

We start with the top auction which has the same properties as in the 2 bidder case: all bidders faces the same threshold $\bar{x}^{t_n}$, the same reservation price, and the good is allocated efficiently to the highest value bidder (provided it is above the reservation value) when all bidders' values are below the threshold x^{t_n}.

Theorem 6 *Suppose that $m_n < \widehat{m}(n)$. The optimal mechanism is the "top auction" in which the bidders have a common threshold $\bar{x}^{t_n}$, i.e., $\bar{x}_1 = ... = \bar{x}_n = \bar{x}^{t_n}$ if and only if the following conditions hold:*

$$\sum_{i=1,...,n} m_i = \bar{x}^{t_n} \frac{1 - F(\bar{x}^{t_n})^n}{1 - F(\bar{x}^{t_n})} - n \int_{r_{t_n}}^{\bar{x}^{t_n}} F(s)^{n-1} ds, \tag{57}$$

$$\frac{m_1 + ... + m_k}{k} - \frac{m_{k+1} + ... + m_n}{n-k} \leq \bar{x}^{t_n} \left(\frac{1 - F(\bar{x}^{t_n})^k}{k(1 - F(\bar{x}^{t_n}))} - \frac{F(\bar{x}^{t_n})^k}{n-k} \frac{1 - F(\bar{x}^{t_n})^{n-k}}{1 - F(\bar{x}^{t_n})} \right) \text{ for } k = 1, ..., n-1, \tag{58}$$

where the reservation value r_{t_n} is uniquely defined by $r_{t_n} = \frac{1-F(r_{t_n}) - \frac{(1-F(\bar{x}^{t_n}))^2}{1-F(\bar{x}^{t_n})+\bar{x}^{t_n} f(\bar{x}^{t_n})}}{f(r_{t_n})}$.

In this mechanism the expected trading probabilities for all $i \in \{1, ..., n\}$ are as follows: $q_i(x_i) = F(x_i)^{n-1}$ for all $x_i \in [r_{t_n}, \bar{x}^{t_n})$, $q_i(x_i) = 0$ for all $x_i \in [0, r_{t_n})$, and $q_i(\bar{x}^{t_n})$ is set to *satisfy i's budget constraint:*

$$m_i = \bar{x}^{t_n} q_i(\bar{x}^{t_n}) - \int_{r_{t_n}}^{\bar{x}^{t_n}} F(s)^{n-1} ds \tag{59}$$

Observe that equation (57) is the aggregate budget constraint. It is obtained by summing up the individual budget constraints (59) and taking into account that $\sum_i q_i(\bar{x}^{t_n}) = \frac{1-F(\bar{x}^{t_n})^n}{1-F(\bar{x}^{t_n})}$, i.e., with probability 1 the good is given to a bidder with value above the threshold $\bar{x}^{t_n}$ if there is at least one such bidder.

Equation (57) defines the common threshold $\bar{x}^{t_n}$ uniquely when $m_1 \leq \widehat{m}(n) \equiv 1 - \int_{r:r-\frac{1-F(r)}{f(r)}=0}^{1} F^{n-1}(x)dx$.[14] At the same (57) and (58) cannot hold simultaneously if $m_1 > \widehat{m}(n)$.[15] So, $m_1 \leq \widehat{m}(n)$ is a necessary condition for the optimality of the top auction.

The family of conditions (58) also has an intuitive economic interpretation. Recall that the budget constraints of all bidders must be binding at the threshold $\bar{x}^{t_n}$. For this to be possible, for all $k \in \{1, ..., n-1\}$, the difference between the average budget of the richest k bidders and the average budget of the poorest $n-k$ bidders cannot exceed the maximal possible difference in average high transfers (transfers paid by bidders with values at least $\bar{x}^{t_n}$) by the bidders in these groups. Conditions (58) state exactly that. Indeed, the maximal average high transfer paid by the k richest bidders is $\bar{x}^{t_n} \frac{1-F(\bar{x}^{t_n})^k}{k(1-F(\bar{x}^{t_n}))} - \int_{r_p}^{\bar{x}^p} F^{n-1}(s)ds$ since $\frac{1-F(\bar{x}^{t_n})^{t_n}}{1-F(\bar{x}^{t_n})}$ is the maximal probability that one of the k richest bidders with values at least $\bar{x}^{t_n}$ gets the good (it reflects that the good is allocated to a member of this group for sure if one of them has value of at least $\bar{x}^{t_n}$). On the other hand, the minimal average high transfer paid by the $n-k$ poorest bidders is $\bar{x}^{t_n} \frac{F(\bar{x}^{t_n})^k(1-F(\bar{x}^{t_n})^{n-k}}{(n-k)(1-F(\bar{x}^{t_n}))} - \int_{r_p}^{\bar{x}^p} F^{n-1}(s)ds$ since $\frac{F(\bar{x}^{t_n})^k(1-F(\bar{x}^{t_n})^{n-k}}{(1-F(\bar{x}^{t_n}))}$ is the minimal probability that one of the $n-k$ poorest bidders with values at least $\bar{x}^{t_n}$ gets the good (it reflects that the good is allocated to a member of this group when one of them has value of at least $\bar{x}^{t_n}$ and the values of all k richest bidders are below $\bar{x}^{t_n}$). So, the difference between these two expressions -the right-hand side of (58)- is the maximal difference between average high transfers of the richest k and poorest $n-k$ bidders.

[14]The solution to (57) is unique because its right-hand side is equal to zero when $\bar{x}^{t_n} = 0$; is equal to $n\widehat{m}(n)$ and so exceeds $\sum_i m_i$ when $\bar{x}^{t_n} = 1$ since $m_1 \leq \widehat{m}(n)$; and is increasing in x^{t_n}. To see the latter note that the derivative of the right-hand side of (57) w.r.t $\bar{x}^{t_n}$ is equal to $\frac{1-F(\bar{x}^{t_n})^n}{1-F(\bar{x}^{t_n})} + \frac{\bar{x}^{t_n} f(\bar{x}^{t_n})}{(1-F(\bar{x}^{t_n}))^2}(1+(n-1)F(\bar{x}^{t_n})^n - nF(\bar{x}^{t_n})^{n-1}) - nF(\bar{x}^{t_n})^{n-1} + nF(r_{t_n})^{n-1}\frac{dr_{t_n}}{d\bar{x}^{t_n}}$, which is positive, in particular, because $\frac{dr_{t_n}}{d\bar{x}^{t_n}} > 0$.

[15]Summing up (57) and the first inequality in (58) for $k=1$ yields $m_1 \leq \bar{x}^{t_n} - \int_{r_{t_n}}^{\bar{x}^{t_n}} F(s)^{n-1}ds$. The right-hand side of this inequality is strictly increasing in $\bar{x}^{t_n}$ on $[0,1]$ and therefore does not exceed $\widehat{m}(n) = 1 - \int_{r:r-\frac{1-F(r)}{f(r)}=0}^{1} F^{n-1}(x)dx$.

Technically, the conditions (57)-(58) are equivalent to the first-order conditions for minimizing the Lagrange dual function $g(\lambda) \equiv \mathcal{L}(\lambda, \bar{x}(\lambda))$ in the case $\lambda_1 = ... = \lambda_n$. Since $g(\lambda)$ is convex, it has a unique minimum characterized by the solution to the first-order conditions, and such solution must also be unique.

So, if conditions (57)-(58) fail to hold,[16] then the bidders cannot have the same threshold in the optimal mechanism. Not all thresholds have to be different: some sets of bidders can form "clusters" with a common threshold. The optimal mechanism in this case -"budget-handicap" auction is characterized in the next Theorem. To state it we need to introduce the following notation. For any set $J \subseteq \{1, ..., n\}$ s.t. $i \notin J$, let $Prob.[\gamma_i(x_i) > \max_{j \in J} \gamma_j] = \prod_{j \in J} \int_{x_j \in [0,1]:\, \gamma_i(x_i) > \gamma_j(x_j)} dF(x_j)$.

Theorem 7 *Suppose that $m_n < \widehat{m}(n)$ and that conditions (57)-(58) fail to hold,. Then the optimal auction is a "budget handicap auction" which is uniquely defined by a vector of thresholds $(\bar{x}_1, ..., \bar{x}_n)$ s.t. $\bar{x}_i \geq \bar{x}_{i+1}$ for all i, with strict inequality for at least some i.*

If $\bar{x}_i > \bar{x}_j$, then $r_i > r_j$ and $q_i(x) < q_j(x)$ for all $x \in [r_j, \bar{x}_j]$.

The optimal profile of threshold values $(\bar{x}_1, ..., \bar{x}_n)$ is unique and is characterized by the following necessary and sufficient conditions:

(i) For bidder $i \geq 2$ such that $\bar{x}_i \neq \bar{x}_j$ for all $j \neq i$, the budget constraint must hold, i.e.:

$$m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s) ds \tag{60}$$

For bidder $i = 1$ when $\bar{x}_1 \neq \bar{x}_j$ for all $j \neq 1$, either the budget constraint (60) holds for $i = 1$, or the following two conditions hold:

$$\bar{x}_1 - \frac{1 - F(\bar{x}_1)}{f(\bar{x}_1)} = \frac{\bar{x}_2^2 f(\bar{x}_2)}{1 - F(\bar{x}_2) + \bar{x}_2 f(\bar{x}_2)}, \tag{61}$$

$$m_1 \geq \bar{x}_1 - \int_0^{\bar{x}_1} q_1(x_1) dx_1, \tag{62}$$

(ii) For bidders $k_1, ..., k_l$ that form a cluster with a common threshold $\bar{x}^c < 1$, i.e., $\bar{x}_{k_1} =$

[16] Intuitively, this happens when the budget inequality between the bidders is sufficiently large that (58) cannot hold.

$... = \bar{x}_{k_l} = \bar{x}^c \neq \bar{x}_j$ for any $j \notin \{k_1, ..., k_l\}$, the following conditions must hold:[17,18]

$$\sum_{h\in\{1,...,l\}} m_{k_h} = \bar{x}^c \frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{i\notin\{k_1,...,k_l\}} \gamma_i] - l\int_0^{\bar{x}^c} q_{k_1}(s)ds \tag{63}$$

$$\text{for all } r \in \{1, ..., l-1\}, \quad \frac{m_{k_1} + ... + m_{k_r}}{r} - \frac{m_{k_{r+1}} + ... + m_{k_l}}{l-r} \leq$$
$$\bar{x}^c \left(\frac{1-F(\bar{x}^c)^r}{r(1-F(\bar{x}^c))} - F(\bar{x}^c)^r \frac{1-F(\bar{x}^c)^{l-r}}{(l-r)(1-F(\bar{x}^c))} \right) Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{i\notin\{k_1,...,k_l\}} \gamma_i] \tag{64}$$

If the common threshold in the cluster $\{k_1, ..., k_l\}$ is $x^c = 1$, then the following condition must hold:

$$m_{k_l} \geq 1 - \int_{r:r-\frac{1-F(r)}{f(r)}=0}^{1} q_{k_l}(s)ds. \tag{65}$$

The main thrust of the Theorem lies in providing necessary and sufficient conditions characterizing the optimal threshold profile. According to Lemma 7, the latter uniquely determines $q_i(x_i)$ for all i and almost all $x_i \in [0, \bar{x}_i)$ and $q_i(\bar{x}_i)$ for all i s.t. $\bar{x}_i \neq \bar{x}_j$ for all $j \neq i$. If i belongs to some cluster with a common threshold $\bar{x}^c$ then, either $q_i(\bar{x}^c) = 1$ if $\bar{x}^c = 1$, or $q_i(\bar{x}^c)$ is uniquely defined via the budget constraint of player i given $\int_0^{\bar{x}_i} q_i(x_i)dx_i$. The conditions (63)-(65) guarantee the feasibility of this choice of $q_i(\bar{x}^c)$.

Condition (60) says that in the optimal mechanism the only necessary and sufficient condition for bidder $i \geq 2$ who does not belong to any cluster (i.e. $\bar{x}_i \neq \bar{x}_j$ for all $i \neq j$) is that her budget constraint is binding at her threshold $\bar{x}_i$. The same is true for bidder 1 when $\bar{x}_1 \neq \bar{x}_j$ for all $j \neq 1$, unless her budget constraint is non-binding, in which case (61) and (62) hold. Note that by Theorem 1, the only bidder whose budget constraint may be non-binding with a threshold strictly below 1 is bidder 1.

When several bidders form a cluster with a common threshold $\bar{x}^c < 1$, then we have two conditions characterizing it. The first is the aggregate budget constraints (63). To understand this condition, note that the probability that the good is given to a bidder from this cluster with valuation above $\bar{x}^c$, $\sum_{r=1,...,l} q_{k_r}(\bar{x}^c)$, is equal to $\frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}} \gamma_j]$. The family of conditions (64) is similar to condition (21) for two bidders

[17] Without loss of generality we assume here that indexes $k_1, ..., k_l$ are ordered according to the budgets, i.e., $k_1 < k_2 ... < k_{l-1} < k_l$ and so $m_{k_1} \geq m_{k_2} ... \geq m_{k_{l-1}} \geq m_{k_l}$. This is so because when (64) holds for this ordering, it also holds for any alternative ordering. Furthermore, if bidders $k_{h_1}, ...k_{h_l}$, $k_{h_1} \leq ... \leq k_{h_l}$ form a cluster with threshold $\bar{x}^c$, then $k_{h_j} = k_{h_1} + j - 1$ for all $h_j \in \{h_1, ..., h_l\}$, i.e., the bidders in a cluster are consecutively ranked in order of their budgets.

[18] Note that by Lemma 4 $q_{k_1}(x) = ...q_{k_l}(x)$ for all x, since bidders $\{k_1, ..., k_l\}$ have the same threshold $\bar{x}^c$ and therefore (by Lemma 5), have the same virtual value functions $\gamma(.)$.

and condition (58) for n bidders in the top auction. It ensures the feasibility of setting the probabilities of trading $(q_{k_1}(\bar{x}^c), ..., q_{k_l}(\bar{x}^c))$ for bidders $k_1, ..., k_l$ in a cluster so that the budget constraints of each of them is binding at the threshold $\bar{x}^c$[19] . It has the same economic content as conditions (58). It says that the difference between the average budget of the richest r bidders and the average budget of the poorest $l-r$ bidders in the cluster cannot exceed the maximal possible difference in average high transfers (transfers paid by bidders with values at least $\bar{x}^c$) by the bidders in these groups. For details, we refer the reader to the discussion of condition (58) that follows the statement of Theorem 6. In particular, because each bidder in the cluster whose value is $\bar{x}^c$ gets the same surplus equal to $\int_0^{\bar{x}^c} q_{k_1}(s)ds$, the right-hand side of (64) is the difference between the maximal average probability that one of the richest r bidders with value at least $\bar{x}^c$ gets the good and the minimal average probability that one of the poorest $l-r$ bidders with value at least $\bar{x}^c$ gets the good. Each of those probabilities is a product of two terms. The second of those terms is $Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j]$, the probability that no bidder outside the cluster has a virtual value exceeding the virtual value of a cluster member of type $\bar{x}^c$, $\gamma_{k_1}(\bar{x}^c)$. The first of those terms is as follows. For the richest r bidders in the cluster it is the average probability that at least one among those r bidders has value of at least $\bar{x}^c$, $\frac{1-F(\bar{x}^c)^r}{r(1-F(\bar{x}^c))}$. For the poorest $l-r$ bidders in the cluster it is the average probability that at least one among $l-r$ bidders has value of at least $\bar{x}^c$ while the values of the other r bidders in the cluster are below $\bar{x}^c$, $F(\bar{x}^c)^r \frac{1-F(\bar{x}^c)^{l-r}}{(l-r)(1-F(\bar{x}^c))}$.

Finally, the condition for the cluster with threshold $\bar{x}^c = 1$ is that each bidder's budget constraint is satisfied. Since each of the bidders in this cluster gets the good with probability 1 when her value is 1, the right-hand side of (65) is the transfer that such type is required to pay. Condition (65) says that the lowest budget in this cluster, m_{k_l}, exceeds this transfer.

The most challenging part in applying this result and computing the optimal "budget handicap" auction is determining which groups of bidders form clusters with common thresholds. Theorem 7 simplifies this task by showing that any cluster contains only "adjacent" bidders with the smallest budget differences. So the number of possible cluster configurations is 2^{n-1}, and potentially one may have to go over all of them to compute the solution. Our results provide a tractable method to check whether a particular cluster configuration is optimal. For example, the optimal mechanism is a budget-handicap auction without any clusters if the following system of n equations has a solution $(\bar{x}_1, ..., \bar{x}_n)$ satisfying $\bar{x}_i > \bar{x}_{i+1}$

[19] See Border (2007) for an analysis of the necessary and sufficient conditions on the expected probabilities of trading for implementation in standard asymmetric auctions.

for all $i \in \{1, ..., n-1\}$:

$$m_1 = \bar{x}_1 - \int_0^{\bar{x}_1} \int_{x_{-1}:\gamma_1(x_1)>\max\{0,\max_{j\neq 1}\gamma_j(x_j)\}} \prod_{j\neq 1} dF(x_j)dx_1$$
$$m_i = \bar{x}_i \int_{x_{-i}:\gamma_i(\bar{x}_i)>\max\{0,\max_{j\neq i}\gamma_j(x_j)\}} \prod_{j\neq i} dF(x_j) - \int_0^{\bar{x}_i} \int_{x_{-i}:\gamma_i(x_i)>\max\{0,\max_{j\neq i}\gamma_j(x_j)\}} \prod_{j\neq i} dF(x_j)dx_i \tag{66}$$

Similarly, we can write down the conditions for the optimality of any other cluster configuration. In the next section of this Appendix we consider an example with three bidders and exhibit conditions for optimality of various cluster configurations in that case.

To conclude this section, we establish the robustness to the number of bidders of the result showing that the seller attains the highest revenue when all bidders have the same budgets and that the seller's revenue does not change after a small redistribution of the budgets across the bidders, but eventually decreases when the budgets become more heterogeneous (in the sense of mean preserving spread) and budget-handicap mechanism becomes optimal.

Theorem 8 *Suppose that the aggregate budget of all bidders is fixed, i.e., $\sum_i m_i = M$.*[20] *Then the seller gets the maximal payoff in the optimal mechanism when all budgets are equal, i.e., $m_i = \frac{M}{n}$ for all $i = 1, ..., n$.*

If the top auction is the optimal mechanism under budget profiles $(m_1, ..., m_n)$ and $(m'_1, ..., m'_n)$ such that $\sum_i m_i = \sum_i m'_i$, then the optimal threshold $\bar{x}^{t_n}$ and the expected seller's revenue is the same under both budget profiles.

If the two budget profiles $(m_1, ..., m_n)$ and $(m'_1, ..., m'_n)$ are such that $\sum_{j=1}^n m_j = \sum_{j=1}^n m'_j$, $\sum_{j=i}^n m_j \leq \sum_{j=i}^n m'_j$ for all $i \in \{2, ..., n\}$ and the optimal mechanism under budget profile $(m_1, ..., m_n)$ is a budget handicap auction, then $\pi(m_1, ..., m_n) < \pi(m'_1, ..., m'_n)$.

2B Examples: Two and Three Bidders under Uniform Type Distribution

2B.1 Two Bidders

In this section we compute the optimal mechanism for two bidders under the uniform type distribution. Let us start with the top auction characterized in Theorem 1. Equation (20)

[20]To make this result non-trivial M has to be sufficiently small. In particular, we will assume that $M \leq n\widehat{m}(n)$.

defining the common threshold $\bar{x}^t$ becomes:

$$m_1 + m_2 = \bar{x}^t + \left(\bar{x}^t\right)^2 - \left(\bar{x}^t\right)^3 + \frac{\left(\bar{x}^t\right)^4}{4}$$

Condition (21) simplifies to $m_1 - m_2 \leq \bar{x}^t(1-\bar{x}^t)$. If this condition holds, then by Theorem 1 the optimal mechanism is a top auction. By equation (16) in Lemma 5, $\lambda^t = (1-\bar{x}^t)^2$ and $\gamma_i(x) = 2x - 2\bar{x}^t + (\bar{x}^t)^2$ for $x \leq \bar{x}^t$. Hence, $r_t = \bar{x}_t - \frac{\bar{x}_t^2}{2}$ and by Lemma 7 $q_i(x_i) = 0$ if $x_i < r_t$; $q_i(x_i) = x_i$ if $x_i \in [r_t, \bar{x}^t)$; $q_i(x_i) = \frac{1+\bar{x}^t}{2} + \frac{m_i - m_j}{2}$ if $x_i \geq \bar{x}^t$.

So, $q_1(x)$ and $q_2(x)$ jump upwards at $x = \bar{x}^t$, except in the borderline case $m_1 - m_2 = \bar{x}^t(1-\bar{x}^t)$ where $q_1(x)$ jumps to 1 at $\bar{x}^t$, and $q_2(x)$ is continuous at $\bar{x}^t$, with $q_2(\bar{x}^t) = F(\bar{x}^t) = \bar{x}^t$.

If $m_1 - m_2 > \bar{x}^t(1-\bar{x}^t)$, then by Theorem 2 the optimal mechanism is a budget-handicap auction with thresholds $\bar{x}_1$ and $\bar{x}_2$ such that $\bar{x}_1 > \bar{x}_2$. Using Theorem 2, $q_1(\bar{x}_1) = 1$, $q_2(\bar{x}_2) = F(\bar{x}_1) = \bar{x}_1$, $r_i = \bar{x}_i - \frac{\bar{x}_2^2}{2}$, $i \in \{1,2\}$, and for $x_i \in [r_i, \bar{x}_i)$:

$$q_i(x_i) = \int_{\gamma_i(x_i) > \gamma_j(s)} ds = \int_{x_i - \bar{x}_i > s - \bar{x}_j} ds = x_i - \bar{x}_i + \bar{x}_j \;\text{ for }\; i, j \in \{1,2\}\; i \neq j. \tag{67}$$

Note that $q_1(x) - q_2(x) = 2(\bar{x}_2 - \bar{x}_1) < 0$ for $x \in [r_1, \bar{x}_2]$, as buyer 1 is handicapped.

To derive $\bar{x}_1$ and $\bar{x}_2$, we need to consider two cases according to Theorem 2. The first case is when (25) and (26) hold. Note that (26) in our uniform distribution case becomes $2\bar{x}_1 - 1 = \bar{x}_2^2$, while (25) can be rewritten using (67) as follows:

$$m_2 = \bar{x}_2\bar{x}_1 - \int_{r_2}^{\bar{x}_2} q_2(x_2)dx_2 = \bar{x}_1\bar{x}_2 - \bar{x}_1\frac{\bar{x}_2^2}{2} + \frac{\bar{x}_2^4}{8} \tag{68}$$

Substituting $2\bar{x}_1 - 1 = \bar{x}_2^2$ into (68) yields $m_2 = \frac{\bar{x}_2 + \bar{x}_2^3}{2} - \frac{\bar{x}_2^2}{4} - \frac{\bar{x}_2^4}{8}$, which has a unique solution $\bar{x}_2(m_2) \in (0,1)$ for all $m_2 \in (0, \widehat{m})$ where $\widehat{m} = \frac{5}{8}$.

By Theorem 2, the thresholds $\bar{x}_2(m_2)$ and $\bar{x}_1(m_2) = \frac{1+\bar{x}_2(m_2)^2}{2}$ characterize the optimal mechanism if (27) holds, which in the case of uniform type distribution becomes:

$$m_1 > \frac{1+\bar{x}_2(m_2)^2}{2} - \frac{\bar{x}_2(m_2)^3}{2} + \frac{\bar{x}_2(m_2)^4}{8}. \tag{69}$$

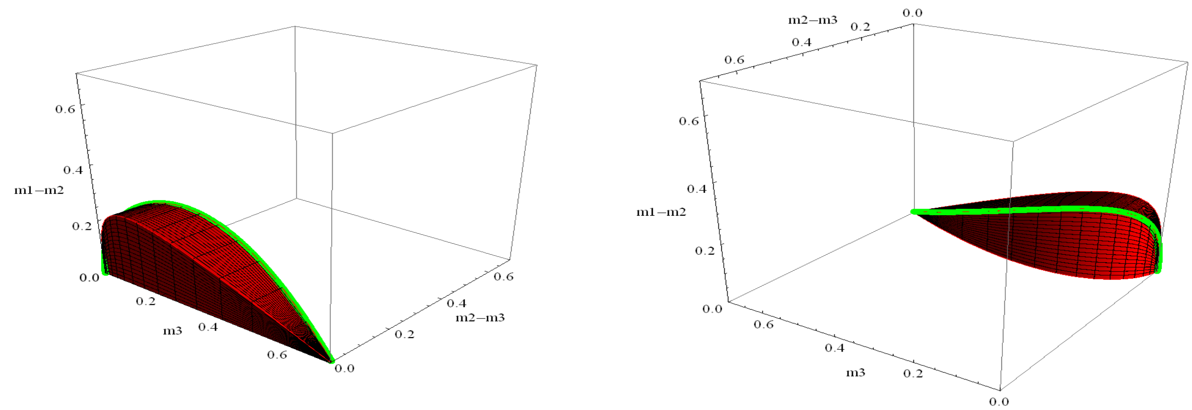
Otherwise, i.e., if (69) does not hold, then the optimal $\bar{x}_1$ and $\bar{x}_2$ are the unique solution to (68) and binding budget constraint of bidder 1, which can be rewritten using (67) as follows:

$$m_1 = \bar{x}_1 - \int_{r_1}^{\bar{x}_1} q_1(x_1)dx_1 = \bar{x}_1 - \frac{\bar{x}_2^3}{2} + \frac{\bar{x}_2^4}{8}$$

2B.2 Three Bidders

The optimal mechanism with three bidders can be of four kinds:

Figure 5: Region of Optimality of The Top Auction



(a) View 1

(b) View 2

- "top-auction:" $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = \bar{x}^t$;
- "budget-handicap auctions" with:
  - "top cluster:" $\bar{x}_1 = \bar{x}_2 > \bar{x}_3$.
  - "lower cluster:" $\bar{x}_1 > \bar{x}_2 = \bar{x}_3$.
  - "no clusters:" $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$.

Interestingly, each of these mechanisms is optimal for a set of budgets of a positive measure, as shown below. To illustrate this and for simplicity we focus on the case in which all budget constraints are binding. By Lemma 9, a sufficient condition for this is that $m_i \leq \frac{1}{2}$.

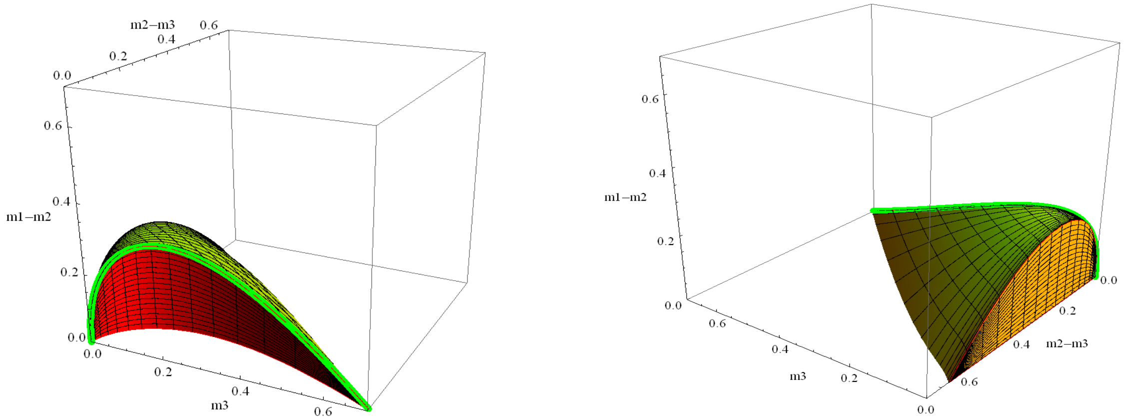
Top Auction

In the top auction, the reservation value is given by $r_t = \bar{x}^t - \frac{\left(\bar{x}^t\right)^2}{2}$. Also, $q_i(x) = x^2$ for all $x \in [r_t, \bar{x}^t)$, and $q_i(\bar{x}^t)$ is set to satisfy the budget constraint of bidder $i \in \{1, 2, 3\}$. Then conditions (63) and (64) simplify to:

$$\begin{aligned} \sum_{i=1}^{3} m_i &= \bar{x}^t(1 + \bar{x}^t) + \left(\bar{x}^t - \frac{\left(\bar{x}^t\right)^2}{2}\right)^3 \\ m_1 - \frac{m_2 + m_3}{2} &\leq \bar{x}^t \left(1 - \bar{x}^t \frac{1 + \bar{x}^t}{2}\right) \\ m_1 - m_3 &\leq \bar{x}^t \left(1 - \left(\bar{x}^t\right)^2\right) \end{aligned} \tag{70}$$

Top auction is optimal when the system (70) has a solution $\bar{x}^t$.

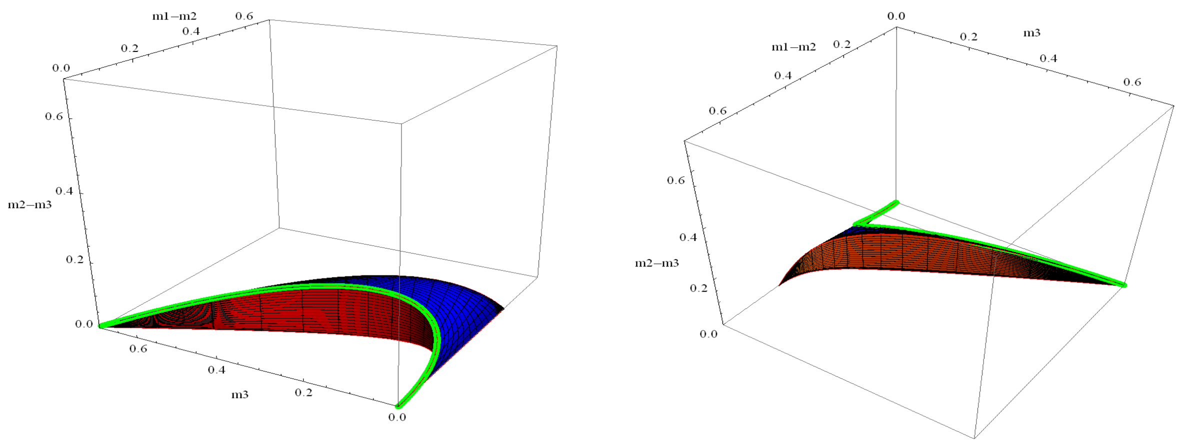
Figure 6: Region of Optimality of the Budget Handicap Auction with Top Cluster



(a) View 1

(b) View 2

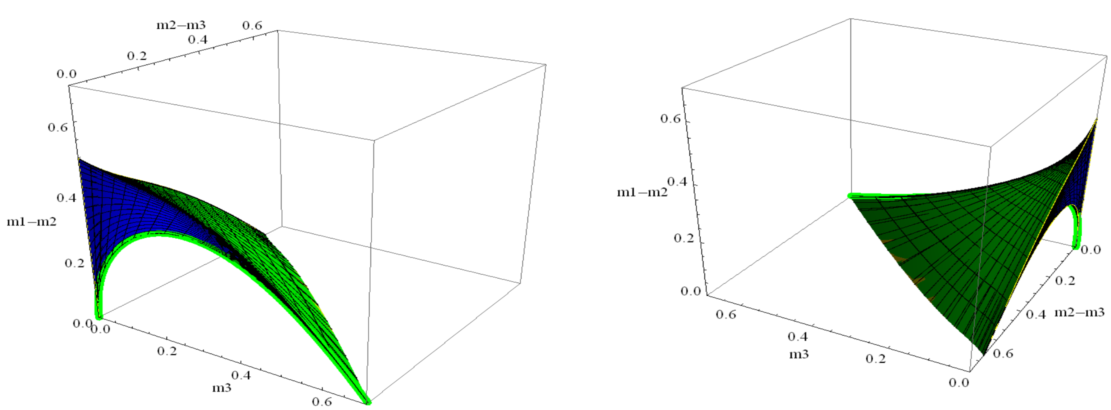
Figure 7: Region of Optimality of the Budget Handicap Auction with Lower Cluster



(a) View 1

(b) View 2

Figure 8: Region of Optimality of the Budget Handicap Auction with No Clusters



(a) View 1

(b) View 2

Budget-Handicap Auction with Top Cluster

Since $\bar{x}_1 = \bar{x}_2$ in the top cluster, we will simplify the notation and let $\bar{x}_1$ denote the threshold of bidders 1 and 2 in the rest of this subsection. So, we have $\bar{x}_1 > \bar{x}_3$, $\gamma_1(x) = \gamma_2(x) = 2x - 2\bar{x}_1 + \bar{x}_1^2$ for $x < \bar{x}_1$, $\gamma_1(\bar{x}_1) = \gamma_2(\bar{x}_1) = \bar{x}_1^2$; $\gamma_3(x) = 2x - 2\bar{x}_3 + \bar{x}_3^2$ for $x < \bar{x}_3$, $\gamma_3(\bar{x}_3) = \bar{x}_3^2$. The bidders' reservation values are given by $r_1 = r_2 = \bar{x}_1 - \frac{\bar{x}_1^2}{2}$, $r_3 = \bar{x}_3 - \frac{\bar{x}_3^2}{2}$.

Then by Lemma 7 for $i \in \{1,2\}$, $q_i(x) = 0$ for $x < \bar{x}_1 - \frac{\bar{x}_1^2}{2}$, $q_i(x) = x(x - \bar{x}_1 + \frac{\bar{x}_1^2}{2} + \bar{x}_3 - \frac{\bar{x}_3^2}{2})$ for $x \in (\bar{x}_1 - \frac{\bar{x}_1^2}{2}, \bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}]$, and $q_i(x) = x$ for $x \in (\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}, \bar{x}_1)$. The values of $q_1(\bar{x}_1)$ and $q_2(\bar{x}_1)$ are determined by the budget constraints of bidders 1 and 2.

For bidder 3, we have $q_3(x) = 0$ for $x < \bar{x}_3 - \frac{\bar{x}_3^2}{2}$, $q_3(x) = \left(x - \bar{x}_3 + \frac{\bar{x}_3^2}{2} + \bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2$ for $x \in (\bar{x}_3 - \frac{\bar{x}_3^2}{2}, \bar{x}_3)$, and $q_3(\bar{x}) = \left(\frac{\bar{x}_3^2}{2} + \bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2$.

Note that while $q_3(x)$ is continuous everywhere above r_3, $q_1(x)$ and $q_2(x)$ experience two jumps. First, there is a jump at $\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}$, as bidders 1 and 2 with values above this level no longer face the competition from bidder 3 because $\gamma_1(\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}) = \gamma_3(\bar{x}_3)$. The second jump happens at the threshold $\bar{x}_1$, since $\lim_{x \to \bar{x}-} q_1(x) + q_2(x) = 2\bar{x} < 1 + \bar{x} = q_1(\bar{x}) + q_2(\bar{x})$.

By Theorem 7, the budget-handicap auction with a top cluster is optimal if the following system of two equations and one inequality has a solution:

$$m_3 = \bar{x}_3 q_3(\bar{x}_3) - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} q_3(x_3) dx_3 \tag{71}$$

$$m_1 + m_2 = (1 + \bar{x}_1) - 2 \int_{\bar{x}_1 - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1} q_1(x_1) dx_1 \tag{72}$$

$$m_1 - m_2 \leq \bar{x}_1(1 - \bar{x}_1).$$

Using the expressions for $q_i(x)$, $i \in \{1,2,3\}$ in (71) and (72) yields:

$$m_3 = \bar{x}_3 \left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} \left(s - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 ds = \bar{x}_3 \left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 - \frac{\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^3}{3} = -\frac{\bar{x}_3^6}{24} + \frac{\bar{x}_3^5}{4} + \bar{x}_3^3 \left(1 - \frac{\bar{x}_3}{4}\right)\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right) + \left(\bar{x}_3 - \frac{\bar{x}_3^2}{2}\right)\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2 \tag{73}$$

$$\begin{aligned} m_1 + m_2 &= \bar{x}_1(1 + \bar{x}_1) - 2\int_{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1} y dy - 2\int_{\bar{x}_1 - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}} y\left(y - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_1^2}{2} - \frac{\bar{x}_3^2}{2}\right)) dy \\ &= \bar{x}_1(1 + \bar{x}_1) + \frac{\bar{x}_3^4}{4}\left(1 - \bar{x}_3 + \frac{\bar{x}_3^2}{6}\right) - \bar{x}_1^3\left(1 - \frac{\bar{x}_1}{4}\right) + \left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)\bar{x}_3^2\left(1 - \frac{\bar{x}_3}{2}\right)^2 \end{aligned} \tag{74}$$

Equations (73) and (74) implicitly define $\bar{x}_1$ and $\bar{x}_3$. If the solution is such that $m_1 - m_2 \leq \bar{x}_1(1-\bar{x}_1)$, then the optimal mechanism is a handicap auction with a "top cluster." The set of budgets for which this holds is depicted in Figure 6.

Budget-Handicap Auction with Lower Cluster

Next, consider the "lower cluster" case with $\bar{x}_1 > \bar{x}_2 = \bar{x}_3$. To simplify the presentation, let $\bar{x}_2$ denote the threshold of bidders 2 and 3 and drop $\bar{x}_3$ from the notation. Then, $\gamma_1(x_1) = 2x - 2\bar{x}_1 + \bar{x}_2^2$ for $x_1 < \bar{x}_1$, $\gamma_1(\bar{x}_1) > \gamma_1^-(\bar{x}_1) = \frac{\bar{x}_2^2}{2}$, $\gamma_2(x) = \gamma_3(x) = 2x - 2\bar{x}_2 + \bar{x}_2^2$ for $x < \bar{x}_2$, $\gamma_2(\bar{x}_2) = \gamma_3(\bar{x}_2) = \bar{x}_2^2$. The reservation values are $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$ and $r_2 = r_3 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$.

The probabilities of trading are given by: $q_1(x_1) = 0$ for $x_1 < \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $q_1(x_1) = (x_1 - \bar{x}_1 + \bar{x}_2)^2$ for $x_1 \in \left[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1\right)$, $q_1(\bar{x}_1) = 1$. For $i \in \{2,3\}$, $q_i(x) = 0$ for $x < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, and $q_i(x) = x\,(x - \bar{x}_2 + \bar{x}_1)$ for $x \in \left[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2\right)$. Finally, $q_2(\bar{x}_2)$ and $q_3(\bar{x}_2)$ are determined by the budget constraints of bidders 2 and 3, correspondingly.

By Theorem 7, condition (60) must hold for bidder 1 and conditions (63) and (64) must hold for bidders 2 and 3, i.e.:

$$m_1 = \bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} (s - \bar{x}_1 + \bar{x}_2)^2\, ds = \bar{x}_1 - \frac{\bar{x}_2^3}{3} + \frac{\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3}$$
$$= \bar{x}_1 - \frac{\bar{x}_2^2}{6}\left(\bar{x}_2^2 + \bar{x}_2\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right) + \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right) = \bar{x}_1 - \frac{\bar{x}_2^4}{2}\left(1 - \frac{\bar{x}_2}{2} + \frac{\bar{x}_2^2}{12}\right) \tag{75}$$

$$m_2 + m_3 = \bar{x}_2\bar{x}_1(1+\bar{x}_2) - 2\int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} s\,(s - \bar{x}_2 + \bar{x}_1)\, ds = \bar{x}_1\bar{x}_2(1+\bar{x}_2) - \frac{2\bar{x}_2^3}{3} + \frac{2\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3}$$
$$- (\bar{x}_1 - \bar{x}_2)\left(\bar{x}_2^2 - \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right) = \bar{x}_1\bar{x}_2(1+\bar{x}_2) + \frac{\bar{x}_2^5}{4}\left(1 - \frac{\bar{x}_2}{3}\right) - \bar{x}_2^3\bar{x}_1\left(1 - \frac{\bar{x}_2}{4}\right) \tag{76}$$

$$m_2 - m_3 \leq \bar{x}_2(1-\bar{x}_2)\bar{x}_1 \tag{77}$$

Equations (75) and (76) implicitly define $\bar{x}_1$ and $\bar{x}_2$. If the solution satisfies (77), the optimal mechanism is the handicap auction with the lower cluster and thresholds $\bar{x}_1$ and $\bar{x}_2 = \bar{x}_3$. The set of budgets for which this is true is depicted in Figure 7.

Budget-Handicap Auction with No Clusters

Finally, we consider the case with no clusters, i.e., $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$.

In this case, $\gamma_1(x_1) = 2x - 2\bar{x}_1 + \bar{x}_2^2$ for $x_1 < \bar{x}_1$, $\gamma_1(\bar{x}_1) > \gamma_1^-(\bar{x}_1) = \frac{\bar{x}_2^2}{2}$, $\gamma_2(x) = 2x - 2\bar{x}_2 + \bar{x}_2^2$ for $x < \bar{x}_2$, $\gamma_2(\bar{x}_2) = \bar{x}_2^2$, $\gamma_3(x) = 2x - 2\bar{x}_3 + \bar{x}_3^3$ for $x < \bar{x}_3$, $\gamma_3(\bar{x}_3) = \bar{x}_3^2$. The reservation values are $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $r_2 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, and $r_3 = \bar{x}_3 - \frac{\bar{x}_3^2}{2}$.

Therefore, the probabilities of trading of bidder 1 are as follows: $q_1(x) = 0$ for $x < \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $q_1(x) = (x - \bar{x}_1 + \bar{x}_2)\left(x - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)$ for $x \in \left[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right]$, $q_1(x) = x - \bar{x}_1 + \bar{x}_2$ for $x \in \left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}, \bar{x}_1\right)$, and $q_1(\bar{x}_1) = 1$.

For bidder 2, $q_2(x) = 0$ for $x < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, $q_2(x) = (x - \bar{x}_2 + \bar{x}_1)\left(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)$ for $x \in \left[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right]$, $q_2(x) = x - \bar{x}_2 + \bar{x}_1$ for $x \in \left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}, \bar{x}_2\right)$, $q_2(\bar{x}_2) = \bar{x}_1$.

Finally, for bidder 3, $q_3(x) = 0$ for $x < \bar{x}_3 - \frac{\bar{x}_3^2}{2}$, $q_3(x) = \left(x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right) \times$ $\times \left(x - \bar{x}_3 + \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)$ for $x \in \left[\bar{x}_3 - \frac{\bar{x}_3^2}{2}, \bar{x}_3\right)$, and $q_3(\bar{x}_3) = (\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})$.

By Theorem 7, in the "no cluster" case the necessary and sufficient conditions characterizing the optimal thresholds $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are the budget constraints (60), i.e., $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_{r_i}^{\bar{x}_i} q_i(s)ds$ for $i = 1, 2, 3$. If the solution to this system of three equations exists and is such that $1 \geq \bar{x}_1 > \bar{x}_2 > \bar{x}_3 \geq 0$, then this configuration with no clusters is optimal.

In the rest of this subsection, we will exhibit the system of three equations $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_{r_i}^{\bar{x}_i} q_i(s)ds$ for $i = 1, 2, 3$ explicitly using the expressions for $q_i(.)$ above and then replace it with a simpler system. First, consider $i = 1$. We have:

$$m_1 = \bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (x - \bar{x}_1 + \bar{x}_2)\left(x - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)dx - \int_{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} x - \bar{x}_1 + \bar{x}_2 ds =$$

$$\bar{x}_1 - \frac{\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_2 - \bar{x}_3 - \frac{\bar{x}_2^2}{2} + \frac{\bar{x}_3^2}{2}\right)}{2}\left(\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2 - \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right)$$

$$-\frac{\bar{x}_2^2}{2} + \frac{\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2}{2} = \bar{x}_1 + \frac{\bar{x}_3^4}{8}\left(1 - \bar{x}_3 + \frac{\bar{x}_3^2}{6}\right) - \frac{\bar{x}_2^3}{2}\left(1 - \frac{\bar{x}_2}{4}\right) + \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)\frac{\bar{x}_3^2}{2}\left(1 - \frac{\bar{x}_3}{2}\right)^2 \tag{78}$$

Second, using the expressions for $q_2(.)$ and $q_3(.)$ derived above, we obtain:

$$m_2 = \bar{x}_2\bar{x}_1 - \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (x - \bar{x}_2 + \bar{x}_1)\left(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)dx - \int_{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} x - \bar{x}_2 + \bar{x}_1 ds \tag{79}$$

$$m_3 = \bar{x}_3(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}) - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} (x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})(x - \bar{x}_3 + \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})dx \tag{80}$$

Next, we replace (79) and (80) with the equations for $m_1 - m_2$ and $m_2 - m_3$ as follows. First, subtracting (79) from (78) we obtain:

$$m_1 - m_2 = \bar{x}_1(1-\bar{x}_2) + \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (\bar{x}_1 - \bar{x}_2)(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2})dx + \int_{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} \bar{x}_1 - \bar{x}_2 ds$$

$$= \bar{x}_1(1-\bar{x}_2) + \frac{\bar{x}_1 - \bar{x}_2}{2}\left(\bar{x}_2^2 - \left(\bar{x}_3 - \frac{\bar{x}_3^2}{2}\right)^2\right). \tag{81}$$

Finally, we perform a change of variable of integration in the second term of (79) to $y = x - \bar{x}_2 + \frac{\bar{x}_2^2}{2} + \bar{x}_3 - \frac{\bar{x}_3^2}{2}$ and subtract (80) from the result to obtain:

$$m_2 - m_3 = \bar{x}_1\bar{x}_2 - \frac{\bar{x}_1^2}{2} + \frac{\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2}{2} - \bar{x}_3\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)$$

$$+ \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} \left(x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2} - \bar{x}_3 + \frac{\bar{x}_3^2}{2}\right)dx =$$

$$\bar{x}_1\bar{x}_2 + (\bar{x}_2\bar{x}_3 - \bar{x}_1(1-\bar{x}_3))\frac{\bar{x}_2^2 - \bar{x}_3^2}{2} + \left(\frac{1}{2} - \bar{x}_3\right)\left(\frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)^2 + \frac{\bar{x}_3^2}{2}(\bar{x}_2 - \frac{\bar{x}_2^2}{2} - \bar{x}_3 + \frac{\bar{x}_3^2}{2})(\bar{x}_1 + \frac{\bar{x}_3^2}{4} - \frac{\bar{x}_2^2}{2}). \tag{82}$$

To conclude, when the solution to the system (78), (81) and (82) satisfies $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$, this is the optimal mechanism. The set of budgets for this case is depicted in Figure 7.

Section 3B. Constrained-Efficient Mechanism under Uniform Type Distribution

In this subsection we compute the constraint-efficient mechanism for two bidders whose types are distributed uniformly on $[0,1]$ and who have budgets m_1 and m_2, respectively.

First, neither budget constraint is binding and the constrained-efficient mechanism is a standard all-pay auction if $m_2 \geq \frac{1}{2}$, since in this case $m_2 \geq 1 - \int_0^1 sds = \frac{1}{2}$.

Now suppose that $m_2 \leq \frac{1}{2}$. Let us first consider top auction. Conditions (34) and (35) yield that the constrained-efficient mechanism is a top auction with threshold $\bar{x}^{te} = m_1 + m_2$ if $m_1 \leq \sqrt{2m_2} - m_2$.

Now suppose that $m_1 > \sqrt{2m_2} - m_2$ and $m_2 \leq \frac{1}{2}$. Then the solution is a budget-handicap auction. First, let us explore the budget-handicap auction with two binding budget constraints. In this case, $\bar{x}_2^e < \bar{x}_1^e < 1$, $\lambda_2^e(x) = \frac{(1-\bar{x}_2^e)^2}{2}$, $\lambda_1^e = -\bar{x}_1^e + \bar{x}_2^e + \frac{(1-\bar{x}_2^e)^2}{2} = -\bar{x}_1^e + \frac{1+(\bar{x}_2^e)^2}{2}$. So $\gamma_2^e(x) = x_2 + \frac{(1-\bar{x}_2^e)^2}{2}$ for $x \leq \bar{x}_2^e$, $\gamma_1^e(x) = x_1 - \bar{x}_1^e + \frac{1+(\bar{x}_2^e)^2}{2}$ for $x < \bar{x}_1^e$, $\gamma_1^e(\bar{x}_1^e) = \frac{1}{2} + \frac{1+\bar{x}_1^e(\bar{x}_1^e - (\bar{x}_2^e)^2)}{2(1-\bar{x}_1^e)} > \frac{1+(\bar{x}_2^e)^2}{2}$. Using Lemma 7 we can now compute the thresholds $\bar{x}_1^e$ and $\bar{x}_2^e$ in the budget-handicap auction:

$$m_1 = \bar{x}_1^e - \int_{x_1 \in [0,\bar{x}_1^e)} q_1(x_1)dx_1 = \bar{x}_1^e - \int_{x_1 \in [0,\bar{x}_1^e)} \int_{\gamma_1^e(x_1) > \gamma_2^e(x_2)} dx_2 dx_1 = \bar{x}_1^e - \frac{(\bar{x}_2^e)^2}{2} \quad (83)$$

$$m_2 = \bar{x}_2^e \bar{x}_1^e - \int_{x_2 \in [0,\bar{x}_2^e)} q_2(x_2)dx_2 = \bar{x}_2^e \bar{x}_1^e - \int_{x_2 \in [0,\bar{x}_2^e)} \int_{\gamma_2^e(x_2) > \gamma_1^e(x_1)} dx_1 dx_2 = \frac{(\bar{x}_2^e)^2}{2} \quad (84)$$

Solving (83) and (84) yield thresholds $\bar{x}_1^e = m_1 + m_2$ and $\bar{x}_2^e = \sqrt{2m_2}$ in the budget-handicap auction when both budget constraints are binding, which is true when $\lambda_1^e = -\bar{x}_1^e + \frac{1+(\bar{x}_2^e)^2}{2} > 0$. This inequality is equivalent to $m_1 < \frac{1}{2}$ given that $\bar{x}_1^e = m_1 + m_2$ and $\bar{x}_2^e = \sqrt{2m_2}$.

Finally, if $m_1 \geq \frac{1}{2} \geq m_2$, then the budget constraint of bidder 1 is no longer binding so (83) does not hold and $\lambda_1 = -\bar{x}_1^e + \frac{1+(\bar{x}_2^e)^2}{2} = 0$. Using the latter equality and (84) we obtain $\bar{x}_1^e = \frac{1}{2} + m_2$, $\bar{x}_2^e = \sqrt{2m_2}$, and the maximal transfer paid by bidder 1 with valuation in $\left[\frac{1}{2} + m_2, 1\right]$ is equal to $\bar{x}_1^e - \frac{(\bar{x}_2^e)^2}{2} = \frac{1}{2}$.

To summarize, the constrained-efficient mechanism in this example is:

(i) A standard symmetric all-pay auction with zero reservation value for each bidder and non-binding budget constraints if $m_2 \geq \frac{1}{2}$.

(ii) Top auction with zero reserve and threshold $x^{te} = m_1 + m_2$ if $m_1 \leq \sqrt{2m_2} - m_2$ and $m_2 \leq \frac{1}{2}$.

(iii) Budget-handicap auction with both budget constraint binding and thresholds $\bar{x}_1^e = m_1 + m_2$ and $\bar{x}_2^e = \sqrt{2m_2}$ if $\sqrt{2m_2} - m_2 < m_1 < \frac{1}{2}$.

(iv) Budget-handicap auction in which only the budget constraint of bidder 2 is binding, with thresholds $\bar{x}_1^e = \frac{1}{2} + m_2$ and $\bar{x}_2^e = \sqrt{2m_2}$, if $m_1 \geq \frac{1}{2} \geq m_2$.

Figure 9a depicts how constrained-efficient mechanism depends on the budgets. Figure 9b highlights budget regions in which the constrained-efficient (listed first) and optimal mechanisms (listed second) are different. Specifically, these differences are as follows:

Area 1: All-pay auction vs. budget- handicap auction with only m_2 binding;

Area 2: Budget-handicap Auction with m_2 binding only vs. budget-handicap auction with both budget constraints binding;

Area 3: All-pay or handicap auction vs. Top auction;

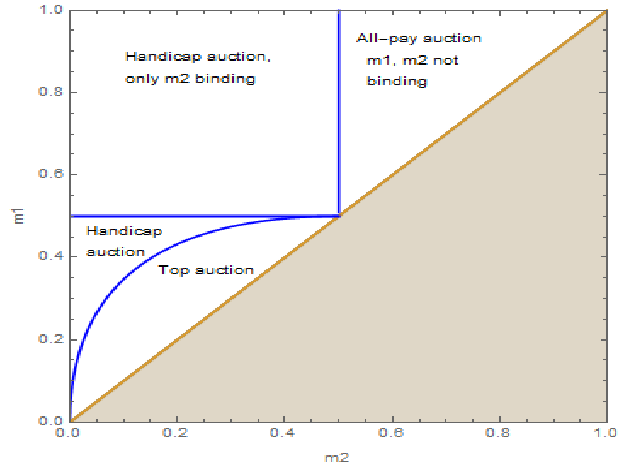
Area 4: Top auction vs Budget-handicap auction;

Area 5: top auction is both constrained-efficient and optimal mechanism.

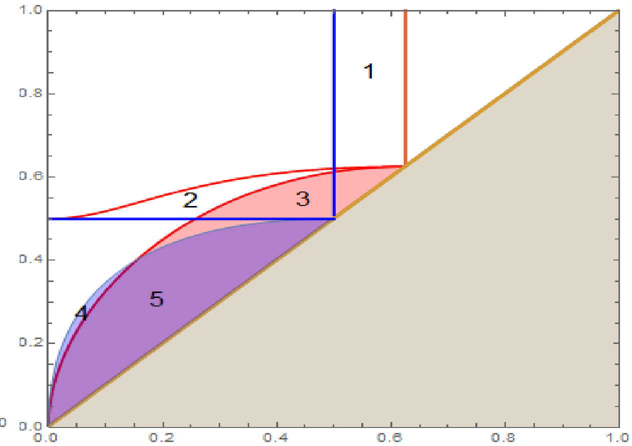
Section 4B. Asymmetrically Distributed Values

In this section we extend our analysis to the case of asymmetrically distributed valuations and show that, for a set of parameter values, the optimal mechanism is a "generalized

Figure 9: Constrained-Efficient and Optimal Mechanisms



(a) Constrained-Efficient Mechanism.



(b) Constrained-Efficient vs Optimal

top auction." In this mechanism, as in the "top auction," bidders with sufficiently high valuations are tied and the good is allocated randomly between them. However, in the generalized top auction the bidders have different valuation thresholds, not a common one as in the top auction, and bidders with the same valuations below the thresholds face different probabilities of trading due to the distribution asymmetry.

For brevity, we will focus on the case of two bidders. Extending the results to n bidders is straightforward but notationally cumbersome. So, suppose that bidder i's, $i \in \{1, 2\}$, valuation is distributed according to probability distribution F_i with increasing hazard rate, and her budget m_i satisfies $m_i - \frac{1-F_i(m_i)}{f_i(m_i)} < 0$. As in the symmetric case, this assumption ensures that budget constraints of all bidders are binding. We do not impose an ordering of m_1 and m_2. However, we make the following ranking assumption.

Assumption 3 *(Monotone Likelihood Ratio) For all $x, x' \in [0, 1]$, $x' > x$, $\frac{f_1(x')}{f_1(x)} > \frac{f_2(x')}{f_2(x)}$.*

Note that Assumption 3 implies that $F_1(.)$ first-order stochastically dominates $F_2(.)$.

A careful perusal of the derivation of the Lagrangian (11), and of the proofs of Lemma 5, which establishes a 1-to-1 relationship between the vectors of thresholds $\bar{x}$ and the Lagrange multipliers λ, and Lemmas 4-6 and 7-10 confirms that all these results apply verbatim to the case of the asymmetric distributions. We omit rewriting these results in order to save space. Next, let us introduce the following definition:

Definition 1 *A generalized top auction is a mechanism in which the bidders' thresholds $\bar{x}_1$,*

$\bar{x}_2$ and expected probabilities of trading $q_1(\bar{x}_1)$ and $q_2(\bar{x}_2)$ satisfy the following conditions:

$$\frac{\bar{x}_1^2 f_1(\bar{x}_1)}{1 - F_1(\bar{x}_1) + \bar{x}_1 f_1(\bar{x}_1)} = \frac{\bar{x}_2^2 f_2(\bar{x}_2)}{1 - F_2(\bar{x}_2) + \bar{x}_2 f_2(\bar{x}_2)}, \tag{85}$$

$$\sum_{i=1,2} (1 - F_i(\bar{x}_i)) q_i(\bar{x}_i) = 1 - F_1(\bar{x}_1) F_2(\bar{x}_2), \tag{86}$$

and in which the probabilities of trading $q_i(x_i)$ for $x_i \in [0, \bar{x}_i)$, $i \in \{1, 2\}$ is uniquely defined by 18 in Lemma 7, with $\gamma_i(x_i) = x_i - \frac{1 - F_i(x_i) - \frac{(1-F_i(\bar{x}_i))^2}{(1 - F_i(\bar{x}_i) + \bar{x}_i f_i(\bar{x}_i))}}{f_i(x_i)}$.

Note that equation (85) says that the buyers' virtual values at the thresholds $\bar{x}_1$ and $\bar{x}_2$, $\gamma_1(\bar{x}_1)$ and $\gamma_2(\bar{x}_2)$, are equal, so it is optimal for the seller to allocate the good randomly across the buyers when $x_1 \geq \bar{x}_1$ and $x_2 \geq \bar{x}_2$. Equation (86) is the feasibility condition on $q_1(\bar{x}_1)$ and $q_2(\bar{x}_2)$ which must be satisfied when the good is allocated to a buyer with a valuation above her threshold iff there is at least on such buyer.

Our proof of the existence and optimality of the generalized top auction will proceed as follows. First, we will substitute $q_i(\bar{x}_i)$ out from (86) by using the budget constraint, i.e., $q_i(\bar{x}_i) = \frac{m_i + \int_0^{\bar{x}_i} q_i(x_i) dx_i}{\bar{x}_i}$, yielding a system of two equations, (85) and modified (86), which only depends on $\bar{x}_1$ and $\bar{x}_2$. Then we will establish that this system has a solution. Besides condition (86), feasibility requires that $q_i(\bar{x}_i) \leq 1$ and $q_i(\bar{x}_i) \geq F_j(\bar{x}_j)$ for $i \in \{1, 2\}$. The latter condition is necessary to ensure that $q_i(.)$ is increasing at $\bar{x}_i$ for $i \in \{1, 2\}$. However, when (86) holds, any two of these feasibility conditions imply the other two. We will use this property in the following Theorem to establish the feasibility of the generalized top auction.

Theorem 9 *(i) There exist $\delta_1, \delta_2 \in (0, 1)$ such that the system of equations (85) and (87) below has a solution $(\bar{x}_1, \bar{x}_2) \in (\delta_1, 1 - \delta_1) \times (\delta_2, 1 - \delta_2)$:*

$$\sum_{i=1,2} \frac{1 - F_i(\bar{x}_i)}{\bar{x}_i} \left(m_i + \int_0^{\bar{x}_i} q_i(x) dx \right) = 1 - F_1(\bar{x}_1) F_2(\bar{x}_2). \tag{87}$$

where $q_i(\cdot)$ are uniquely defined in (18) in Lemma 7.

(ii) There exists $\epsilon > 0$ s.t. whenever $|F_2(x) - F_1(x)| < 0$ for $x \in [0, 1]$ and $|m_2 - m_1| \leq \epsilon$, then then the solution $(\bar{x}_1, \bar{x}_2) \in (0, 1)^2$ to (85) and (87) is unique and satisfies the feasibility conditions $F_j(\bar{x}_j) \leq \frac{m_i + \int_0^{\bar{x}_i} q_i(x_i) dx_i}{\bar{x}_i} \leq 1$ for $i, j \in \{1, 2\}$.

The optimal mechanism is a generalized top auction with these thresholds, $(\bar{x}_1, \bar{x}_2)$.

Proof: Note that (87) is obtained by substituting $q_i(\bar{x}_i)$ from (86) using the budget constraints of each bidder, $m_i = \bar{x}_i q_1(\bar{x}_1) - \int_0^{\bar{x}_i} q_i(x_i) dx_i$. Thus, the thresholds $(\bar{x}_1, \bar{x}_2)$ in a generalized top auction must satisfy (85)and (87). Claims 1-3 below establish that a solution to this system of two equations exists.

After establishing this, we will need to verify that our solution $(\bar{x}_1, \bar{x}_2)$ to (85)and (87) is such that $q_1(\bar{x}_1) = \frac{m_1 + \int_0^{\bar{x}_1} q_1(x_1)dx_1}{\bar{x}_1}$ and $q_2(\bar{x}_2) = \frac{m_2 + \int_0^{\bar{x}_2} q_2(x_1)dx_1}{\bar{x}_2}$ are feasible, i.e., $F_j(\bar{x}_j) \leq q_i(\bar{x}_i) \leq 1$. Claim 4 establishes that these feasibility conditions hold under the conditions of the Theorem. Finally, Claim 5 completes the proof by establishing the uniqueness of the solution.

Claim 1. *Equation (85) defines a continuous, increasing and one-to-one mapping $\bar{x}_2(\bar{x}_1)$ between $\bar{x}_1 \in [0,1]$ and $\bar{x}_2 \in [0,1]$ such that $\bar{x}_2(1) = 1$ and $\bar{x}_2(0) = 0$.*

Proof of Claim 1: Note that $\frac{\bar{x}_i^2 f_i(\bar{x}_i)}{1-F_i(\bar{x}_i)+\bar{x}_i f_i(\bar{x}_i)}$, $i = 1, 2.$ is continuous and increasing in $\bar{x}_i$ on $[0, 1]$ by the increasing hazard rate assumption, and is equal to 0 (1) if $\bar{x}_i = 0$ ($\bar{x}_i = 1$), from which Claim 1 follows immediately.

Claim 2. *The mapping $\bar{x}_2(\bar{x}_1)$ defined by (85) is such that $\bar{x}_2(\bar{x}_1) \leq \bar{x}_1$, and $\frac{d\bar{x}_2}{d\bar{x}_1}_{\bar{x}_1=1} = 1$.*

Proof of Claim 2: Let us rewrite (85) as follows:

$$\frac{1}{\bar{x}_1} + \frac{1 - F_1(\bar{x}_1)}{\bar{x}_1^2 f_1(\bar{x}_1)} = \frac{1}{\bar{x}_2} + \frac{1 - F_2(\bar{x}_2)}{\bar{x}_j^2 f_2(\bar{x}_2)}. \tag{88}$$

By the increasing hazard rate assumption, the left-hand (right-hand) side of (88) is monotonically decreasing in $\bar{x}_1$ ($\bar{x}_2$).

Further, since $\frac{1-F_1(x)}{f_1(x)} \geq \frac{1-F_2(x)}{f_2(x)}$, the left-hand side of (88) is greater than its right-hand-side at any $\bar{x}_1 = \bar{x}_2$. Hence, (88) holds as equality only if $\bar{x}_1 \geq \bar{x}_2$.

Next, differentiating (88) yields:

$$\left(-\frac{2}{\bar{x}_1^2} - \frac{f_1'(\bar{x}_1)(1 - F_1(\bar{x}_1))}{\bar{x}_1^2 f_1(\bar{x}_1)} - \frac{2(1 - F_1(\bar{x}_1))}{\bar{x}_1^3 f_1(\bar{x}_1)}\right) = \left(-\frac{2}{\bar{x}_2^2} - \frac{f_2'(\bar{x}_2)(1 - F_2(\bar{x}_2))}{\bar{x}_2^2 f_1(\bar{x}_2)} - \frac{2(1 - F_2(\bar{x}_2))}{\bar{x}_2^3 f_2(\bar{x}_2)}\right)\frac{d\bar{x}_2}{d\bar{x}_1}. \tag{89}$$

From (89) and $\bar{x}_2(1) = 1$ it follows that $\frac{d\bar{x}_2}{d\bar{x}_1}_{\bar{x}_1=1} = 1$.

Claim 3. *The system of equations (85), (87) has a solution $(\bar{x}_1, \bar{x}_2)$. Any such solution belongs to $(\delta_1, 1 - \delta_1) \times (\delta_2, 1 - \delta_2)$ for some $\delta_1, \delta_2 \in (0, 1)$.*

Proof of Claim 3: Using the mapping $\bar{x}_2(\bar{x}_1)$ defined by equation (85) and described in Claims 1 and 2, we can rewrite equation (87) as follows: $G_1(\bar{x}_1) = G_2(\bar{x}_1)$ where

$$G_1(\bar{x}_1) = \frac{1 - F_1(\bar{x}_1)}{\bar{x}_1} m_1 + \frac{1 - F_2(\bar{x}_2(\bar{x}_1))}{\bar{x}_2(\bar{x}_1)} m_2 \tag{90}$$

$$G_2(\bar{x}_1) = 1 - F_1(\bar{x}_1)F_2(\bar{x}_2(\bar{x}_1)) - \frac{1 - F_1(\bar{x}_1)}{\bar{x}_1}\int_0^{\bar{x}_1} q_1(x_1)\,dx_1 - \frac{1 - F_2(\bar{x}_2(\bar{x}_1))}{\bar{x}_2(\bar{x}_1)}\int_0^{\bar{x}_2(\bar{x}_1)} q_2(x_2)\,dx_2 \tag{91}$$

Differentiating (90) and (91) and using $\bar{x}_2(1) = 1$ and $\left(\frac{d\bar{x}_2}{d\bar{x}_1}\right)_{\bar{x}_1=1} = 1$ yields:

$$G_1'(1) = -\sum_{i=1,2} m_i f_i(1), \tag{92}$$

$$G_2'(1) = -\sum_{i=1,2} f_i(1)\left(1 - \int_0^1 q_i(x_i)\, dx_i\right). \tag{93}$$

By Lemma 7, when $\bar{x}_1 = 1$ then $q_i(x_i) = 0$ for $x_i < r_i(1)$ where $r_i(1)$ is defined by $r_i(1) - \frac{1-F_i(r_i(1))}{f_i(r_i(1))} = 0$. So, $1 - \int_0^1 q_i(x_i)dx_i \geq r_i(1)$. But since $m_i - \frac{1-F_i(m_i)}{f_i(m_i)} < 0$ for $i \in \{1,2\}$ by assumption, we have $m_i < r_i(1) \leq 1 - \int_0^1 q_i(x_i)dx_i$. So, from (92) and (93) it follows that $0 > G_1'(1) > G_2'(1)$. But since $G_1(1) = G_2(1) = 0$, it follows that there exists $\delta' > 0$ s.t. $G_1(x_1) < G_2(x_1)$ for all $x_1 \in [1-\delta', 1]$.

On the other hand, $G_1(.)$ is monotonically decreasing on $[0,1]$, $\lim_{x_1 \to 0} G_1(x_1) \to \infty$ and $G_2(0) = 1$. So, there exists $\delta'' > 0$ s.t. $G_1(x_1) > G_2(x_1)$ if $x_1 \in [0, \delta'']$. Let $\delta_1 = \min\{\delta', \delta''\}$. So, since $G_1(.)$ and $G_2(.)$ are continuous, there exists $\bar{x}_1 \in (\delta_1, 1-\delta)$ such that $G_1(\bar{x}_1) = G_2(\bar{x}_1)$. Then $\bar{x}_1$ and $\bar{x}_2 = \bar{x}_2(\bar{x}_1)$ constitute a solution to the system (85), (87). Since the mapping $\bar{x}_2(\bar{x}_1)$ is continuous and satisfies $\bar{x}_2(0) = 0$ and $\bar{x}_2(\bar{x}_1) \leq \bar{x}_1$ by Claim 2, we have $\bar{x}_2 = \bar{x}_2(\bar{x}_1) \in (0,1)$. So, with a slight abuse of notation, from now on let $(\bar{x}_1, \bar{x}_2)$ denote such solution.

Next, we set $q_i(\bar{x}_i) = \frac{m_i + \int_0^{\bar{x}_i} q_i(x_i)dx_i}{\bar{x}_i}$ for $i \in \{1,2\}$. So (87) can be rewritten as follows:

$$q_1(\bar{x}_1)(1 - F_1(\bar{x}_1)) + q_2(\bar{x}_2)(1 - F_2(\bar{x}_2)) = 1 - F_1(\bar{x}_1)F_2(\bar{x}_2)). \tag{94}$$

Claim 4. *A solution $(\bar{x}_1, \bar{x}_2)$ to (85) and (87) satisfies the feasibility conditions $F_j(\bar{x}_j) \leq q_i(\bar{x}_i) \leq 1$ for $i, j \in \{1,2\}$ if and only if the inequalities (95) and (96) hold.*

$$m_1 - m_2 \leq \bar{x}_1 - \bar{x}_2 F_1(\bar{x}_1) - \int_0^{\bar{x}_1} q_1(x_1)\, dx_1 + \int_0^{\bar{x}_2} q_2(x_2)\, dx_2, \tag{95}$$

$$m_1 - m_2 \geq \bar{x}_1 F_2(\bar{x}_2) - \bar{x}_2 - \int_0^{\bar{x}_1} q_1(x_1)\, dx_1 + \int_0^{\bar{x}_2} q_2(x_2)\, dx_2, \tag{96}$$

The **"Only If"** part of the claim is obvious. If the feasibility conditions $F_j(\bar{x}_j) \leq q_i(\bar{x}_i) \leq 1$ for $i, j \in \{1,2\}$ hold, then using these conditions in the budget constraints $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(x_i)dx_i$ yields (95) and (96).

In the opposite direction, note that from (94) it follows immediately that $F_j(\bar{x}_j) \leq q_i(\bar{x}_i)$ if and only if $q_j(\bar{x}_j) \leq 1$ for $i, j \in \{1,2\}$. So, if $q_1(\bar{x}_1) > 1$ then $q_2(\bar{x}_2) < F_1(\bar{x}_1)$. Using these inequalities in the budget constraints yields that (95) fails. A similar argument shows that if $q_2(\bar{x}_2) > 1$ then by (94) $q_1(\bar{x}_1) < F_2(\bar{x}_2)$, and so (96) fails.

Claim 5. There exists $\eta > 0$ such that (95) and (96) hold if $|F_2(x) - F_1(x)| < \eta$ for all $x \in [0,1]$ and $|m_1 - m_2| < \eta$.

Proof: First, we need to introduce some notation. Let r_i be the unique solution for x_i to $\gamma_i(x_i) = 0$ for $i \in \{1,2\}$. Then for $x_i \in [r_i, \bar{x}_i]$ define $\hat{x}_j(x_i)$ as a solution for x_j to the equation $\gamma_i(x_i) = \gamma_j(x_j)$. That is, $\hat{x}_2(x_1)$ ($\hat{x}_1(x_2)$) is the solution in x_2 (x_1) to the following equation:

$$x_1 - \frac{1 - F_1(x_1) - \frac{(1-F_1(\bar{x}_1))^2}{(1-F_1(\bar{x}_1)+\bar{x}_1 f_1(\bar{x}_1))}}{f_1(x_1)} = x_2 - \frac{1 - F_2(x_2) - \frac{(1-F_2(\bar{x}_2))^2}{(1-F_2(\bar{x}_2)+\bar{x}_2 f_2(\bar{x}_2))}}{f_2(x_2)}. \tag{97}$$

Note that both $\hat{x}_1(.)$ and $\hat{x}_2(.)$ are increasing, continuous, and satisfy $\widehat{x}_i(r_j) = r_i$ and $\widehat{x}_i(\bar{x}_j) = \bar{x}_i$ for $i, j \in \{1,2\}$.

Further, let us show that $\hat{x}_2(x_1) < x_1$ for all $x_1 \in (r_1, \bar{x}_1]$. Since $\gamma_i'(x) > 0$, it is sufficient to establish that $\gamma_2(x) > \gamma_1(x)$ for all $x \in [r_2, \bar{x}_2]$.

First, since $\gamma_i(x)$ is continuous in x for $i \in \{1,2\}$, $\gamma_1(\bar{x}_1) = \gamma_2(\bar{x}_2)$ and, as established above, $\bar{x}_1 > \bar{x}_2$, it follows that there exists $\eta > 0$ s.t. $\gamma_2(x) > \gamma_1(x)$ for all $x \in [\bar{x}_2 - \eta, \bar{x}_2]$. So, if $\gamma_2(x) \le \gamma_1(x)$ for some $x \in [r_2, \bar{x}_2)$, there exists $\tilde{x} \in [r_2, \bar{x}_2)$ s.t. $\gamma_2(\tilde{x}) = \gamma_1(\tilde{x})$ and $\gamma_2'(\tilde{x}) > \gamma_1'(\tilde{x})$ which, by definition of $\gamma_i(.)$, implies that $\frac{f_2'(\tilde{x})}{f_2(\tilde{x})} \ge \frac{f_1'(\tilde{x})}{f_1(\tilde{x})}$. However, the last inequality contradicts Assumption 3 (MLRP). Hence, we must have $\gamma_2(x) > \gamma_1(x)$ for all $x \in [r_2, \bar{x}_2)$ and therefore $\hat{x}_2(x_1) < x_1$ for all $x_1 \in [r_1, \bar{x}_1]$.

Using this notation, we have $q_i(x_i) = F_j(\hat{x}_j(x_i))$ if $x_i \ge r_i$ and $q_i(x_i) = 0$ otherwise.

Our next step is to prove a lower bound for the right-hand sides of (95) and an upper bound for the right-hand side of (96). For this, we need to bound the expression $\int_0^{\bar{x}_1} q_1(x)dx - \int_0^{\bar{x}_2} q_2(x)dx$. We have: $\int_0^{\bar{x}_2} q_2(x_2)dx_2 =$

$$\begin{aligned}
&\int_{r_2}^{\bar{x}_2} F_1(\hat{x}_1(x_2))dx_2 = \int_0^{\bar{x}_1} \bar{x}_2 - \max\{r_2, \hat{x}_2(x_1)\}dF_1(x_1) = \bar{x}_2 F_1(\bar{x}_1) - r_2 F_1(r_1) - \int_{r_1}^{\bar{x}_1} \hat{x}_2(x_1)dF_1(x_1) \\
&\ge \bar{x}_2 F_1(\bar{x}_1) - r_2 F_1(r_1) - \int_{r_1}^{\bar{x}_1} x_1 dF_1(x_1) = (\bar{x}_2 - \bar{x}_1)F_1(\bar{x}_1) + (r_1 - r_2)F_1(r_1) + \int_{r_1}^{\bar{x}_1} F_1(x_1)dx_1,
\end{aligned} \tag{98}$$

where the first equality has been established above, the second equality is obtained by changing the order of integration, the inequality holds because $\hat{x}_2(x_1) \le x_1$, and the last equality is obtained by integrating by parts. Combining (98) with $\int_0^{\bar{x}_1} q_1(x_1) = \int_{r_1}^{\bar{x}_1} F_2(\hat{x}_2(x_1))dx_1 \le \int_{r_1}^{\bar{x}_1} F_2(x_1)dx_1$ yields the following lower bound for the right-hand side of (95):

$$\bar{x}_1(1 - F_1(\bar{x}_1)) + (r_1 - r_2)F_1(r_1) - \int_{r_1}^{\bar{x}_1} F_2(x_1) - F_1(x_1)dx_1 \tag{99}$$

Since $r_1 > r_2$, $\bar{x}_1 (1 - F_1(\bar{x}_1)) + (r_1 - r_2)F_1(r_1) > 0$. So, there exists $\epsilon > 0$ s.t. (99) and hence the right-hand side of (95) is positive when $|F_2(x) - F_1(x)| < \epsilon$.

Next, let us provide an upper bound for the right-hand side of (96) and show that this upper bound is negative under the conditions of the Theorem. First, we have:

$$\int_0^{\bar{x}_1} q_1(x_1)dx_1 = \int_{r_1}^{\bar{x}_1} F_2(\hat{x}_1(x_2))dx_1 = \int_0^{\bar{x}_2} (\bar{x}_1 - \max\{\widehat{x}_{12}(x_2), r_1\})\, dF_2(x_2) =$$
$$\bar{x}_1 F_2(\bar{x}_2) - r_1 F_2(r_2) - \int_{r_2}^{\bar{x}_2} \widehat{x}_{12}(x_2) dF_2(x_2) \geq \bar{x}_1 F_2(\bar{x}_2) - r_1 F_2(r_2) - \bar{x}_1(F_2(\bar{x}_2) - F_2(r_2)), \tag{100}$$

where the first equality has been established above, the second equality is obtained by changing the order of integration, the inequality holds because $\hat{x}_1(x_2) \leq \bar{x}_1$ for all $x_2 \in [0, \bar{x}_2]$, and the last equality is obtained by integrating by parts.

Combining (100) with $\int_0^{\bar{x}_2} q_2(x_2) = \int_{r_2}^{\bar{x}_2} F_1(\hat{x}_1(x_2))dx_2 \leq F_1(\bar{x}_1)(\bar{x}_2 - r_2)$ yields the following upper bound for the right-hand side of (96):

$$-\bar{x}_2(1 - F_1(\bar{x}_1)) + r_1 F_2(r_2) + \bar{x}_1(F_2(\bar{x}_2) - F_2(r_2)) - F_1(\bar{x}_1) r_2 \tag{101}$$

From (85) and (87) it is easy to see that there exist constants $K_1 > 0$ and $K_2 > 0$ s.t. $|\bar{x}_1 - \bar{x}_2| < \epsilon K_1$ and $|\bar{r}_2 - \bar{r}_1| < \epsilon K_2$ if $|m_2 - m_1| < \epsilon$ and $|F_2(x) - F_1(x)| < \epsilon$ for all $x \in [0, 1]$. So when $\epsilon > 0$ is sufficiently small then (101), and hence the right-hand side of (96) are negative.

So, when the right-hand side of (95) is positive and the right-hand side of (96) is negative, both (95) and (96) holds when $|m_1 - m_2| < \psi$ when is sufficiently small. So, setting $\eta = \min\{\epsilon, \psi\}$ concludes the proof of Claim 5. *Q.E.D.*